THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 29, 1898. Thirty-six persons were in attendance at the two sessions, including the following twenty-nine members of the Society:—Professor Maxime Bôcher, Professor A. S. Chessin, Dr. J. B. Chittenden, Professor F. N. Cole, Professor T. S. Fiske, Mr. G. B. Germann, Miss Ida Griffiths, Dr. G. W. Hill, Professor Harold Jacoby, Mr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Professor James McMahon, Mr. James Maclay, Professor E. H. Moore, Professor Frank Morley, Professor Simon Newcomb, Mr. J. C. Pfister, Professor James Pierpont, Professor M. I. Pupin, Professor J. K. Rees, Dr. Frank Schlesinger, Professor C. A. Scott, Mr. W. M. Strong, Professor H. D. Thompson, Dr. Jacob Westlund, Professor M. W. Whitney, Miss E. C. Williams, and Professor R. S. Woodward.

The President of the Society, Professor Simon Newcomb, occupied the chair during the two sessions. The Council announced the election of the following persons to membership in the Society:—Mr. Edward B. Escott, Grand Rapids, Mich.; Dr. Loring B. Mullen, Central High School, Cleveland, Ohio; Professor James Mills Peirce, Harvard University, Cambridge, Mass.; Professor Alexander Pell, University of South Dakota, Vermillion, S. D.; Professor Arthur Ranum, University of Washington, Seattle, Wash.; Mr. Alfred North Whitehead, Trinity College, Cambridge, Eng.; Mr. Walter C. Wright, Medford, Mass. Five applications for membership were received.

The following papers were presented:

(1) Professor F. MORLEY: "A regular configuration of ten line pairs in hyperbolic space."

(2) Professor R. S. WOODWARD: "The mutual gravitational attraction of two bodies whose mass distributions are symmetrical with respect to the same axis."

(3) Professor E. D. ROE: "On symmetric functions."

(4) Professor A. S. CHESSIN: "Note on the problem of three bodies."

(5) Professor MAXIME BÔCHER: "On singular points of linear differential equations with real coefficients."

(6) Professor E. O. LOVETT: "Contact transformations of developable surfaces."

(7) Dr. L. E. DICKSON: "The largest linear homogeneous group with an invariant Pfaffian."

In the absence of the authors, the papers of Professor Roe, Professor Lovett, and Dr. Dickson were read by title.

Abstracts of those papers which are not intended for publication in the BULLETIN are given below.

Professor Woodward's paper deals with certain problems in the theory of attraction which, although fairly accessible to treatment, seem to have been overlooked hitherto. Let m and m' denote any two masses having mass distributions symmetrical with respect to the same axis. Their gravitational potential is*

$$V = k\int\int \frac{dmdm'}{s} \tag{1}$$

where k is the gravitational constant and s is the distance between dm and dm'. Let the rectangular coördinates of dm and dm' be x, y, z and x', y', z' respectively, and the polar coördinates r, φ, λ and r', φ', λ' respectively, φ, φ' denoting latitudes measured from the xy-plane, and λ, λ' longitudes measured from the xz-plane. Then

$$\begin{aligned} s^2 &= (x-x')^2 + (y-y')^2 + (z-z')^2 \\ &= r^2 + r'^2 - 2rr'\cos\gamma, \\ \cos\gamma &= \sin\varphi\sin\varphi' + \cos\varphi\cos\varphi'\cos(\lambda-\lambda'). \end{aligned} \tag{2}$$

Let the axis of z be the axis of symmetry and let the desired attraction be denoted by A. Then

$$\begin{aligned} A = \frac{\partial V}{\partial z} &= \frac{\partial V}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial z} \\ &= \frac{\partial V}{\partial r}\sin\varphi + \frac{\partial V}{\partial \varphi}\frac{\cos\varphi}{r} = -\frac{\partial V}{\partial z'} \\ &= -\left(\frac{\partial V}{\partial r'}\sin\varphi' + \frac{\partial V}{\partial \varphi'}\frac{\cos\varphi'}{r'}\right), \end{aligned} \tag{3}$$

wherein the derivatives with respect to r, φ, r', φ' are to be found by operating on $1/s$.

* If potential is defined as in (1), then the usual form, $V = k\int dm/s$, should be called potential per unit mass.

Now it is well known that

$$\frac{1}{s} = \frac{1}{r}\left(P_0 + P_1\frac{r'}{r} + P_2\frac{r'^2}{r^2} + \cdots\right) \text{ for } r > r',$$
$$\frac{1}{s} = \frac{1}{r'}\left(P_0 + P_1\frac{r}{r'} + P_2\frac{r^2}{r'^2} + \cdots\right) \text{ for } r < r', \tag{4}$$

in which $P_0, P_1, \cdots$ are Legendre's polynomials, or spherical harmonics. It is well known also that P_n has the form $Q_n Q_n' +$ terms multiplied by cosines of multiples of $(\lambda - \lambda')$. Hence, for the case of symmetry here considered, the only part of P_n that need be retained is $Q_n Q_n'$, since terms multiplied by cosines of $(\lambda - \lambda')$ will vanish in the integrations of (3) with respect to λ and λ', each of which has the limits 0 and 2π. Q_n and Q_n' are zonal harmonic functions of φ and φ'. They are most conveniently expressed by the formula of Rodrigues. Thus, writing for brevity

$$\xi = \sin\varphi, \quad \xi' = \sin\varphi', \tag{5}$$

Q_n is given by

$$Q_n = \frac{1}{2^n n!}\frac{d^n}{d\xi^n}(\xi^2 - 1)^n. \tag{6}$$

Operating on the first of (4) as specified by (3), there results

$$A = -k\iint\frac{dmdm'}{r^2}\Sigma\, Q_n'\left\{(n+1)\,Q_n\sin\varphi - \frac{dQ_n}{d\varphi}\cos\varphi\right\}\left(\frac{r'}{r}\right)^n$$
$$= -k\iint\frac{dmdm'}{r^2}\Sigma\, Q_n\left\{nQ_n'\sin\varphi' + \frac{dQ_n'}{d\varphi'}\cos\varphi'\right\}\left(\frac{r'}{r}\right)^n.$$

These expressions are simplified by the following relations, which are easily proved by means of (6), namely:*

$$(n+1)\,Q_n\sin\varphi - \frac{dQ_n}{d\varphi}\cos\varphi = (n+1)\,Q_{n+1},$$
$$nQ_n\sin\varphi + \frac{dQ_n}{d\varphi}\cos\varphi = nQ_{n-1}. \tag{7}$$

* The first of these equations is given by Todhunter, History of the Theories of Attraction and Figure of the Earth, vol. 2, p. 102. The sum of (7) is a familiar expression.

Making use of these relations the attraction becomes

$$A = -k \int\int \frac{dmdm'}{r^2} \sum_{n=0}^{n=\infty} (n+1) Q_{n+1} Q_n \left(\frac{r'}{r}\right)^n \qquad (8)$$

Denoting the densities of the element masses by ρ and ρ' respectively,

$$dm = \rho r^2 dr \cos\varphi d\varphi d\lambda,$$
$$dm' = \rho' r'^2 dr' \cos\varphi' d\varphi' d\lambda'.$$

Introducing these values and making use of the abbreviations (5), equation (8) becomes

$$A = -\frac{4}{3} k\pi^2 \int\int\int\int \Big(Q_1 Q_0' + 2 Q_2 Q_1' \frac{r'}{r} + 3 Q_3 Q_2' \frac{r'^2}{r^2} + \cdots \Big) \rho\rho' dr dr' d\xi d\xi'. \qquad (9)$$

The corresponding expression coming from the second of (4) is found by interchanging r with r' and Q_n with Q'_{n-1} in (9). The problem is thus reduced to one of integration with respect to the four variables r, r', ξ, ξ', the densities and ρ' being in general functions of those variables.

In Professor Roe's paper symmetric functions are considered from two quite different standpoints. Part I deals with symmetric functions as a whole, and is critical and expository. Two new methods for finding symmetric functions as a whole are given, one of which depends upon the use of Aronhold's operator applied to the resultant of two binary forms, and the solution of the identical equations resulting therefrom; the other is based upon the theorem of corresponding matrices, and was stated to the author by Professor Gordan, of Erlangen. Part II treats of the isolation of the individual terms of a symmetric function together with their coefficients. The coefficient of

$$a_0^{\lambda_0} a_1^{\lambda_1} \cdots a_n^{\lambda_n} \text{ in } a_0^m \sum a_1^{\kappa_1} a_2^{\kappa_2} \cdots a_n^{\kappa_n}$$

is denoted by $\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \cdots n^{\lambda_n} \\ 0^m x_1 x_2 \cdots x_n \end{pmatrix}$

Forms having

$$\lambda_0 > 0, \quad \lambda_n > 0, \quad x_1 = m, \quad x_n = 0,$$

are called normal forms; all other forms, the completely re-

ducible forms excepted, may be reduced to such as are normal forms having a lower m or n. Formulas are given for the four kinds of reduction for passing from higher to lower forms, and also for the four kinds of derivation for passing from lower to higher forms. The completely reducible form has the general formula

$$\begin{pmatrix} 1^{\kappa_1-\kappa_2} 2^{\kappa_2-\kappa_3} \cdots n^{\kappa_n} \\ 0^{\kappa_1} z_1^{\lambda_2} \cdots z_n \end{pmatrix}$$

and is equal to $(-1)^{\kappa_1+\kappa_2+\cdots+\kappa_n}$,

while the normal form

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n} \\ 0^m m^{\mu_0} (m-1)^{\mu_1} \cdots 0^{\mu_m} \end{pmatrix}$$

is given as the sum of simpler forms, by the formula,

$$\mu_0 x \begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n} \\ 0^m m^{\mu_0} (m-1)^{\mu_1} \cdots 0^{\mu_m} \end{pmatrix} =$$

$$-\sum_{r=1}^{r=n} (1+\mu_r) x \begin{pmatrix} 0^{\lambda_0+1} 1^{\lambda_1} 2^{\lambda_2} \cdots r^{\lambda_r-1} \cdots n^{\lambda_n} \\ 0^m m^{\mu_0-1} (m-1)^{\mu_1} \cdots (m-r)^{\mu_r+1} \cdots 0^{\mu_m} \end{pmatrix}$$

In a paper presented at the summer meeting of the AMERICAN MATHEMATICAL SOCIETY, Professor Chessin showed that relative motion of a system may be considered as if it were absolute motion disturbed by space motion, and the author gave several expressions for what he called the *perturbative function of relative motion*. In Professor Chessin's present paper this theory is applied to a case of the problem of three bodies, m_1, m_2, m_3, namely when the mass of one of them (m_3) is infinitely small compared to the masses of m_1 and m_2, while the eccentricity of the orbits of m_1 and m_2 is zero, so that m_1 and m_2 move uniformly in concentric circles about their common center of gravity. Such is very nearly the case of a small planet in the presence of the sun and Jupiter. Such is, in particular, very approximately the case of satellites. For the case considered the undisturbed motion is that of a body (m_3) attracted to two fixed points (m_1 and m_2). This problem can be solved by means of elliptic functions. If λ and μ denote the elliptic coördinates of m_3 and φ the angle which the plane through m_1, m_2, and m_3 forms with its initial posi-

tion, then we can express λ, μ, φ, and t as functions of one parameter (θ). The disturbed motion is obtained by assuming that the masses m_1 and m_2 are now revolving about their common center of gravity with the uniform angular velocity (ω). The perturbative function of relative motion is

$$\Omega = \omega \cos \varphi \left\{ f \frac{d\Delta}{dt} + \left(\frac{\lambda\mu}{c}\right)^2 \frac{d}{dt}\left(\frac{c\Delta}{\lambda\mu}\right) \right\}$$

where c denotes half the distance between m_1 and m_2; Δ the distance of m_3 from the line joining m_1 and m_2

$$\Delta = \frac{1}{c}\sqrt{(\lambda^2 - c^2)(c^2 - \mu^2)};$$

and f is a coefficient depending on the masses m_1 and m_2 and reducing to zero when these are equal. While the problem can not be solved rigorously this method affords a solution by approximation where the convergence of all the developments is established *a priori*.

In Lie's theory of contact transformations we have the theorem demonstrated by Lie, Darboux, and Mayer that the relations

$$[X_i, X_j] = [X_i, Z] = [P_i, P_j] = 0,$$

$$[P_i, Z] = \rho P_i, \qquad [P_i, X_j] \begin{matrix} = 0 \\ = \rho \end{matrix} \begin{pmatrix} i \neq j, \\ i = j, \ \rho \neq 0 \end{pmatrix}$$

where
$$[F_i, F_j] = \sum_1^n \left(\frac{\partial F_i}{\partial p_k}\frac{dF_j}{dx_k} - \frac{\partial F_j}{\partial p_k}\frac{dF_i}{dx_k} \right),$$

$$\frac{dF_j}{dx_k} = \frac{\partial F_j}{\partial x_k} + p_k \frac{\partial F_j}{\partial z} = (j, k)$$

are the necessary and sufficient conditions that the $2n + 1$ equations

$$z' = Z(z, x_1, \cdots, x_n, p_1, \cdots, p_n), \quad x_i' = X_i, p_i' = P_i \ (i = 1, \cdots, n)$$

define a contact transformation. The problem of Professor Lovett's paper is to determine the contact transformations which leave invariant the partial differential equation

$$|p_{11}, p_{22}, \cdots, p_{nn}| = 0.$$

By putting

$$F^{(i)} = F_{x_i} + p_i F_z + \sum_1^n{}_j p_{ij} F_{p_j} \quad p_i = \frac{\partial z}{\partial x_i}, \quad p_{ij} = \frac{\partial^2 z}{\partial x_i \partial x_j},$$
$$(i = 1, \cdots, n)$$

and $$\Phi(X_1, X_2, X_3, \cdots, X_n) = |X_1^{(1)}, X_2^{(2)}, \cdots, X_n^{(n)}|,$$

we find $$|P_{11}, P_{22}, \cdots, P_{nn}| = \frac{\Phi(P_1, P_2, \cdots, P_n)}{\Phi(X_1, X_2, \cdots, X_n)},$$

in virtue of the equations

$$dZ = \sum_1^n{}_i P_i dX_i, \quad dP_i = \sum_1^n{}_j P_{ij} dX_j.$$

By developing $\Phi(P_1, \cdots, P_n)$ and equating the coefficients of $p_{11}, p_{12}, \cdots$ to zero, after an easy reduction the initial conditions give the complete systems

$$(i, 1) = 0, \quad (i, 2) = 0, \cdots, \quad (i, n) = 0; \quad (i = 1, \cdots, n).$$

The application of the method of Mayer to these systems gives

$$P_i = \varphi_i\left(\sum_1^n{}_j p_j x_j - z, p_1, \cdots, p_n\right) = \varphi_i(\zeta, p_1, \cdots, p_n)$$
$$(i = 1, \cdots, n)$$

where the functions φ_i are arbitrary. We verify easily that

$$Z = \sum_1^n{}_i \varphi_i X_i - \varphi_0(\zeta, p_1, \cdots, p_n)$$

in which φ_0 is arbitrary. Letting

$$M_1 = (\psi_1, \cdots, \psi_n), \quad \cdots, M_{n+1} = (\psi_1, \cdots, \psi_n)_{n+1}$$

be the minors of the m's in the determinant

$$\begin{vmatrix} \psi_{1\zeta} & \psi_{2\zeta} & \cdots, & \psi_{n\zeta} & m_1 \\ \psi_{1p_1} & \psi_{2p_1} & \cdots, & \psi_{np_1} & m_2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \psi_{1p_n} & \psi_{2p_n} & \cdots, & \psi_{np_n} & m_{n+1} \end{vmatrix}$$

we have finally

$$X_i = \frac{(\varphi_{i+1}, \cdots, \varphi_n, \varphi_0, \cdots, \varphi_{i-1})_1 - \sum_1^n{}_j x_j(\varphi_{i+1} \cdots, \varphi_n, \varphi_0, \cdots, \varphi_{i-1})_{j+1}}{\sum_1^n{}_j x_j(\varphi_1, \varphi_2, \cdots, \varphi_n)_{j+1} - (\varphi_1, \varphi_2, \cdots, \varphi_n)_1}.$$

If C be the contact transformation whose defining functions are the above X_i, P_i, Z; Q an arbitrary point transformation; and L the transformation of Legendre as generalized by Lie it may be shown analytically and geometrically that

$$C = LQL.$$

In case the contact transformations degenerate into point transformations, Q must be projective. Among the results of the note are complete generalizations of those of a memoir of G. Vivanti, *Rend. del circ. mat. di Palermo*, vol. 5 (1891).

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CONCERNING A LINEAR HOMOGENEOUS GROUP IN $C_{m,q}$ VARIABLES ISOMORPHIC TO THE GENERAL LINEAR HOMOGENEOUS GROUP IN m VARIABLES.

BY DR. L. E. DICKSON.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 20, 1898.)

1. While the present paper is concerned chiefly with continuous groups, its results may be readily utilized for discontinuous groups.* Indeed, the finite form of the general transformation of the group is known *ab initio*. Further, the method is applicable to the construction of a linear $C_{m,q}$-ary group isomorphic to an arbitrary m-ary linear group.

2. The formula of composition of m-ary linear homogeneous substitutions

$$(a_{ij}): \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (j = 1, \cdots, m)$$

is as follows, where the matrix (a_{ij}') operates first:

$$(a_{ij}'') = (a_{ij})\,(a_{ij}'),$$

where

$$a_{ij}'' = \sum_{k=1}^{m} a_{ik}a_{kj}' \qquad (i, j = 1, \cdots, m).$$

* An analogous isomorphism between certain linear groups in the Galois field of order p^n has been discussed by the writer in an article presented to the London Mathematical Society.

Using Sylvester's *umbral* notation, consider the qth minors of the determinant $|a_{ij}|$

$$\begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_q j_1} & a_{i_q j_2} & \cdots & a_{i_q j_q} \end{vmatrix} \equiv \begin{vmatrix} i_1 & i_2 \cdots & i_q \\ j_1 & j_2 \cdots & j_q \end{vmatrix} a .$$

The formula* expressing the qth minors of $|a_{ij}''|$ in terms of the qth minors of $|a_{ij}|$ and of $|a_{ij}'|$ is as follows:

$$(1) \qquad \begin{vmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{vmatrix} a'' = \sum_{l_1, \cdots, l_q} \begin{vmatrix} i_1 \cdots i_q \\ l_1 \cdots l_q \end{vmatrix} a \cdot \begin{vmatrix} l_1 \cdots l_q \\ j_1 \cdots j_q \end{vmatrix} a'$$

the summation extending over the $C_{m,q}$ combinations $l_1, l_2, \cdots, l_q$ of the m integers $1, 2, \cdots, m$ taken q at a time.

3. Consider the $C_{m,q}$-ary linear substitution

$$[a]: \qquad Y_{i_1 i_2 \cdots i_q} = \sum_{l_1, \cdots, l_q} \begin{vmatrix} i_1 & i_2 \cdots & i_q \\ l_1 & l_2 \cdots & l_q \end{vmatrix} a \; Y_{l_1 l_2 \cdots l_q},$$

where the sets $(i_1, \cdots, i_q)$ and $(l_1, \cdots, l_q)$ take successively the $C_{m,q}$ combinations of the integers $1, 2, \cdots, m$ taken q together and where further

$$i_1 < i_2 < \cdots < i_q; \quad l_1 < l_2 < \cdots < l_q.$$

Its determinant has been called the qth compound of the determinant

$$\begin{vmatrix} 1 & 2 \cdots & m \\ 1 & 2 \cdots & m \end{vmatrix} a$$

and equals† the latter raised to the power $C_{m-1, q-1}$.

In virtue of (1) we have the composition formula:

$$[a] \cdot [a'] = [a''].$$

Hence, if the substitutions (a) form a group, so do also the substitutions $[a]$. We will speak of the latter group as the "qth compound of the m-ary group." Hence the theorem:

An arbitrary linear group is isomorphic to each of its compounds.

4. Consider the more general substitution

$$[a]_\epsilon: \qquad X'_{i_1 \cdots i_q} = \sum_{l_1, \cdots, l_q} \varepsilon^{i_1 \cdots i_q}_{l_1 \cdots l_q} \begin{vmatrix} i_1 & \cdots & i_q \\ l_1 & \cdots & l_q \end{vmatrix} a \; X_{l_1 \cdots l_q},$$

* Scott, Theory of Determinants, p. 53.

† Muir, Theory of Determinants, § 174.

where the ϵ's denote ± 1. The product $[a]_\epsilon \cdot [a']_\epsilon$ equals

$$X'_{i_1 \cdots i_q} = \sum_{j_1, \cdots, j_q} \{\epsilon^{i_1 \cdots i_q}_{i_1 \cdots i_q} \epsilon^{j_1 \cdots j_q}_{j_1 \cdots j_q} | {}^{i_1 \cdots i_q}_{i_1 \cdots i_q} |_a \, | {}^{j_1 \cdots j_q}_{j_1 \cdots j_q} |_{a'} X_{j_1 \cdots j_q}\}.$$

Hence if we define the ϵ's such that

$$\epsilon^{i_1 \cdots i_q}_{i_1 \cdots i_q} \cdot \epsilon^{j_1 \cdots j_q}_{j_1 \cdots j_q} = \epsilon^{i_1 \cdots i_q}_{j_1 \cdots j_q} \tag{2}$$

we have the formula of composition

$$[a]_\epsilon \cdot [a']_\epsilon = [a'']_\epsilon.$$

But $[a]_\epsilon = 1$ will correspond to $(a) = 1$ if and only if

$$\epsilon^{i_1 \cdots i_q}_{i_1 \cdots i_q} = +1 \qquad (^{i_1, \cdots, i_q = 1, \cdots, m}_{i_1 < \cdots < i_q}). \tag{3}$$

From (2) and (3) it follows that

$$\epsilon^{i_1 \cdots i_q}_{i_1 \cdots i_q} = \epsilon^{i_1 \cdots i_q}_{i_1 \cdots i_q}. \tag{4}$$

Hence if we set

$$Y_{i_1 \cdots i_q} \equiv \epsilon^{i_1 \cdots i_q}_{1 \cdots q} X_{i_1 \cdots i_q},$$

it follows from (2) and (4) that $[a]_\epsilon$ takes the form $[a]$ of § 3. Since $[a]$ is the transformed of $[a]_\epsilon$ by a linear substitution, their determinants are equal.

We confine our discussion to the group of the $[a]$. Denote the general m-ary linear group by G_m and its qth compound by $C_{m,q}$.

Infinitesimal Transformations of $C_{m,q}$, §§ 5–7.

5. Consider first the case $m = 4$, $q = 2$. Setting

$$a_{jj} = 1 + a_{jj}\delta t, \quad a_{ij} = a_{ij}\delta t, \tag{5}$$

the general infinitesimal transformation of $C_{4,2}$ is seen to assign to the six variables $Y_{i_1 i_2}$ the following increments:

	$Y_{12}\delta t$	$Y_{13}\delta t$	$Y_{14}\delta t$	$Y_{23}\delta t$	$Y_{24}\delta t$	$Y_{34}\delta t$
δY_{12}	$a_{11}+a_{22}$	a_{23}	a_{24}	$-a_{13}$	$-a_{14}$	0
δY_{13}	a_{32}	$a_{11}+a_{33}$	a_{34}	a_{12}	0	$-a_{14}$
δY_{14}	a_{42}	a_{43}	$a_{11}+a_{44}$	0	a_{12}	a_{13}
δY_{23}	$-a_{31}$	a_{21}	0	$a_{22}+a_{33}$	a_{34}	$-a_{24}$
δY_{24}	$-a_{41}$	0	a_{21}	a_{43}	$a_{22}+a_{44}$	a_{23}
δY_{34}	0	$-a_{41}$	a_{31}	$-a_{42}$	a_{32}	$a_{33}+a_{44}$

Setting in turn one of the a_{ij} equal unity and the other 15 equal zero, we obtain 16 linearly independent infinitesimal transformations A_{ij}. These we exhibit (by detached coefficients) in sets of four each. We use the abbreviation

$$P_{ij} \equiv \frac{\partial f}{\partial Y_{ij}} \partial t.$$

Set (1)

	P_{12}	P_{13}	P_{14}
A_{11}	Y_{12}	Y_{13}	Y_{14}
A_{12}	0	Y_{23}	Y_{24}
A_{13}	$-Y_{23}$	0	Y_{34}
A_{14}	$-Y_{24}$	$-Y_{34}$	0

Set (2)

	P_{13}	P_{23}	P_{34}
A_{22}	Y_{12}	Y_{23}	Y_{24}
A_{21}	0	Y_{13}	Y_{14}
A_{23}	Y_{13}	0	Y_{34}
A_{24}	Y_{14}	$-Y_{34}$	0

Set (3)

	P_{13}	P_{23}	P_{34}
A_{33}	Y_{13}	Y_{23}	Y_{34}
A_{31}	0	$-Y_{12}$	Y_{14}
A_{32}	Y_{12}	0	Y_{24}
A_{34}	Y_{14}	Y_{24}	0

Set (4)

	P_{14}	P_{24}	P_{34}
A_{44}	Y_{14}	Y_{24}	Y_{34}
A_{41}	0	$-Y_{12}$	$-Y_{13}$
A_{42}	Y_{12}	0	$-Y_{23}$
A_{43}	Y_{13}	Y_{23}	0

The four transformations of each set generate a group of four parameters. Indeed A_{ii} is Euler's homogeneous operator for the variables of the ith set, which do not enter into the coefficients of the other three of that set, so that the latter are commutative. Thus, for set (1), we have the commutator relations

$$(A_{ij}A_{11}) = A_{ij} \ (j = 2, 3, 4); \quad (A_{ij}A_{ik}) = 0 \quad (j, k = 2, 3, 4).$$

Its invariants are found by expanding the four determinants of the third order, one of which is skew-symmetric and therefore zero. The other three give the function (Pfaffian)

$$F \equiv Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}$$

multiplied by Y_{23}, Y_{24}, Y_{34} respectively.

A similar result holds for the other sets. A skew-symmetric determinant appears in set (2) if we change the signs in the first column, in set (3) if we change the signs in the first and second columns. It is seen that F is an invariant for the total group of 16 parameters. We obtain also the (here trivial) invariant system formed by the six variables $Y_{i_1 i_2}$.

6. Consider the case of general m and q. Neglecting

terms having the factor δt^2, as will be proven allowable, we have at once

$$\begin{vmatrix} i_1 i_2 \cdots i_q \\ i_1 i_2 \cdots i_q \end{vmatrix}_\alpha = 1 + (a_{i_1 i_1} + \cdots + a_{i_q i_q})\delta t ;$$

$$\begin{vmatrix} i_1 i_2 \cdots i_q \\ j_1 j_2 \cdots j_q \end{vmatrix}_\alpha = 0,$$

if two or more j's differ from every i.

Consider the case in which $j_1, j_2, \cdots, j_{s-1}, j_{s+1}, \cdots, j_q$ form a permutation of $i_1, i_2, \cdots, i_{r-1}, i_{r+1}, \cdots, i_q$, while $j_s \neq i_r$. Since

$$i_k < i_{k+1}, \quad j_k < j_{k+1} \qquad (k = 1, \cdots, q-1),$$

the above permutation must be cyclic. According as $s < r$ or $s > r$, we readily see that

$$\begin{vmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{vmatrix}_\alpha$$

must be of the respective forms:

$$\begin{vmatrix} i_1 \cdots i_{s-1}\, i_s\, i_{s+1} \cdots i_{r-1}\, i_r \quad i_{r+1} \cdots i_q \\ i_1 \cdots i_{s-1}\, j_s\, i_s \quad \cdots i_{r-2}\, i_{r-1} \quad i_{r+1} \cdots i_q \end{vmatrix}_\alpha,$$

$$\begin{vmatrix} i_1 \cdots i_{r-1}\, i_r \quad i_{r+1} \cdots i_{s-1}\, i_s\, i_{s+1} \cdots i_q \\ i_1 \cdots i_{r-1}\, i_{r+1}\, i_{r+2} \cdots i_s \quad j_s\, i_{s+1} \cdots i_q \end{vmatrix}_\alpha,$$

the cylic permutation being confined to the i's which run from i_r to i_r inclusive, and of the backward or forward type according as $s \gtrless r$. As the two cases are really not distinct, we consider only the first one, $r > s$.

Substituting for the a_{ij} their values in terms of the $a_{ij}\delta t$, the first determinant takes the following form (where for the moment a_{ik} is written for $a_{i_i i_k}\delta t$ and j for j_s):

$$\begin{array}{|cccccccc|} 1+a_{11} & a_{12} & \cdots & a_{1\,s-1} & a_{1j} & a_{1s} & a_{1\,s+1} & a_{1\,r-1} & a_{1q} \\ a_{21} & 1+a_{22} & \cdots & a_{2\,s-1} & a_{2j} & a_{2s} & a_{2\,s+1} & a_{2\,r-1} & a_{2q} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{s-1\,1} & a_{s-1\,2} & \cdots & 1+a_{s-1\,s-1} & a_{s-1\,j} & a_{s-1\,s} & a_{s-1\,s+1} & a_{s-1\,r-1} & a_{s-1\,q} \\ a_{s1} & a_{s2} & \cdots & a_{s\,s-1} & a_{sj} & 1+a_{ss} & a_{s\,s+1} & a_{s\,r-1} & a_{sq} \\ a_{s+1\,1} & a_{s+1\,2} & \cdots & a_{s+1\,s-1} & a_{s+1\,j} & a_{s+1\,s} & 1+a_{s+1\,s+1} & a_{s+1\,r-1} & a_{s+1\,q} \\ \hline a_{r-1\,1} & a_{r-1\,2} & \cdots & a_{r-1\,s-1} & a_{r-1\,j} & a_{r-1\,s} & a_{r-1\,s+1} & 1+a_{r-1\,r-1} & a_{r-1\,q} \\ a_{r1} & a_{r2} & \cdots & a_{r\,s-1} & a_{rj} & a_{rs} & a_{r\,s+1} & a_{r\,r-1} & a_{rq} \\ \hline a_{q1} & a_{q2} & \cdots & a_{q\,s-1} & a_{qj} & a_{qs} & a_{q\,s+1} & a_{q\,r-1} & 1+a_{qq} \end{array}$$

In the expansion of this determinant, the only term of the first degree in the a's is seen to be a_{ij}. Hence the determinant equals

$$(-1)^{r-s} a_{i_s j_s} \delta t.$$

Similarly, the second determinant is found to have the same value.

The general infinitesimal transformation of the form $[a]$ is therefore as follows:

$$\delta Y_{i_1 \cdots i_q} \equiv Y'_{i_1 \cdots i_q} - Y_{i_1 \cdots i_q}$$

$$= \delta t \{ (a_{i_1 i_1} + \cdots + a_{i_q i_q}) Y_{i_1 \cdots i_q}$$

$$+ \sum_{r,s}^{1 \cdots q} (-1)^{r+s} a_{i_s j_s} Y_{i_1 \cdots i_{s-1} j_s i_{s+1} \cdots i_{r-1} i_{r+1} \cdots i_q} \},$$

the summation also extending over all values of j_s from i_{s-1} to i_s exclusive. A simplification arises by introducing several coexistent notations for the same variable Y, viz:

$$Y_{i_1 \cdots i_{s-1} j_s i_{s+1} \cdots i_q} \equiv (-1)^{s-1} Y_{j_s i_1 \cdots i_{s-1} i_{s+1} \cdots i_q}.$$

Indeed, we may then perform the above summation with respect to s, and obtain for $\delta Y_{i_1 \cdots i_q}$ the simpler value

$$\delta t \{ (a_{i_1 i_1} + \cdots + a_{i_q i_q}) Y_{i_1 \cdots i_q}$$

$$+ \sum_{r,j} (-1)^{r-1} a_{i_r j} Y_{j i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} \},$$

$$(r = 1, \cdots, q; \quad j = 1, \cdots, m; \quad j \neq i_1, i_2, \cdots, i_q).$$

7. We may now readily obtain m^2 linearly independent infinitesimal transformations A_{ik} by setting in turn $a_{ik} = 1$ and the other a's equal zero.

For A_{ij}, $\delta Y_{i_1 \cdots i_q}$ is zero unless one of the i's equals l, while

$$\delta Y_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} = Y_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}.$$

Hence

$$\delta Y_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} = Y_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}.$$

Hence A_{ii} has the form given below (for $k = l$).

For $A_{ik}(l \neq k)$, $\delta Y_{i_1 \cdots i_q}$ is zero unless some $i_r = l$, in which case

$$\delta Y_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} = \sum_j (-1)^{r-1} a_{ij} Y_{j i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}.$$

Hence, since $a_{ij} = 0$ if $j \neq k$,

$$\delta Y_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} = Y_{i_1 \cdots i_{r-1} k i_{r+1} \cdots i_q}.$$

The m^2 independent transformations of the group $C_{m,q}$ are thus:

$$A_{ik} \equiv \sum_{i_1, \cdots, i_{r-1} i_{r+1}, \cdots, i_q}^{1, \cdots, m} Y_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_q k} P_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_q l}.$$

$$(i_1 < i_2 < \cdots < i_q, \text{ and each } \neq l, \neq k).$$

Here $P_{i \cdots i_q}$ denotes

$$\frac{\partial f}{\partial Y_{i \cdots i_q}} \delta t.$$

CERTAIN PROPERTIES OF THE INVARIANTS OF $C_{m,2}$, §§ 8–10.

8. For $q = 2$, we have the m^2 transformations of $C_{m,2}$

$$A_{ik} \equiv \sum_{i \neq k, i}^{i=1, \cdots, m} Y_{ki} P_{il} \qquad (l, k = 1, \cdots, m).$$

We may separate these m^2 transformations into m sets

$$[A_{1l}, A_{2l}, \cdots, A_{ml}] \qquad (l = 1, \cdots, m).$$

Those of the lth set involve only the $m - 1$ differential coefficients

$$P_{1l}, P_{2l}, \cdots, P_{l\,l-1}, P_{l\,l+1}, \cdots, P_{ml}.$$

For use below we exhibit them in a table (with detached coefficients). By our notation $Y_{ij} \equiv - Y_{ji}$.

	P_{1l}	P_{2l}	P_{3l}	$P_{l\,l-1}$	$P_{l\,l+1}$	P_{ml}
A_{1l}	Y_{11}	Y_{12}	Y_{13}	$Y_{1\,l-1}$	$Y_{1\,l+1}$	Y_{1m}
A_{2l}	Y_{21}	0	Y_{23}	$Y_{2\,l-1}$	$Y_{2\,l+1}$	Y_{2m}
A_{3l}	Y_{31}	Y_{32}	0	$Y_{3\,l-1}$	$Y_{3\,l+1}$	Y_{3m}
$A_{l-1\,l}$	$Y_{l-1\,1}$	$Y_{l-1\,2}$	$Y_{l-1\,3}$	0	$Y_{l-1\,l+1}$	$Y_{l-1\,m}$
$A_{l+1\,l}$	$Y_{l+1\,1}$	$Y_{l+1\,2}$	$Y_{l+1\,3}$	$Y_{l+1\,l-1}$	0	$Y_{l+1\,m}$
A_{ml}	Y_{m1}	Y_{m2}	Y_{m3}	$Y_{m\,l-1}$	$Y_{m\,l+1}$	0

It follows exactly as in § 5 that the m transformations of any set generate a group of m parameters.

Deleting the row A_{ij}, we obtain a skew-symmetric determinant of order $m-1$, which we denote by $D_{ij}^{(m-1)}$. Deleting the row A_{ik} and moving the column headed by P_{ik} into the place of the last column, we obtain a bordered skew-symmetric determinant $D_{ik}^{(m-1)}$, the first row and the last column forming its borders.

9. For m odd and $q=2$, we have*

$$D_{ij}^{(m-1)} = [1, 2, \cdots, l-1, l+1, \cdots, m]^2,$$

where the Pfaffian $[1, 2, \cdots, l-1, l+1, \cdots, m]$ includes the extreme cases $[1, 2, \cdots, m-1]$ and $[2, 3, \cdots, m]$. Further $D_{ik}^{(m-1)}$ factors into two Pfaffians of like order, which are seen to be

$$[l, i_1, i_2, \cdots, i_{m-3}], \ [i_1, i_2, \cdots, i_{m-3}, k],$$

where $i_1 < i_2 < \cdots < i_{m-3}$, and each $i \neq l, \neq k$.

Since the interchange of two indices merely changes the sign of the Pfaffian, it follows that all the determinants $D_{ik}^{(m-1)}$ vanish if and only if the Pfaffians

$$F_1 \equiv [2, 3, \cdots, m], \cdots, F_l \equiv [1, 2, \cdots, l-1, l+1, \cdots, m],$$

$$\cdots, F_m \equiv [1, 2, \cdots, m-1]$$

simultaneously vanish. It follows, therefore, from the general theory of Lie that every system of equations invariant under the group $C_{m,2}$, m odd, must include the m equations

$$F_k = 0 \qquad (k = 1, \cdots, m).$$

It follows readily from the properties of Pfaffians that the transformations A_{ij} have the following effect upon the Pfaffians F_k:

$$A_{ik}(F_k) = 0, \quad A_{il}(F_k) = F_k \delta t \qquad (l = 1, \cdots, m; l \neq k),$$

$$A_{ij}(F_k) = 0, \quad A_{il}(F_k) = (-1)^{l+k-1} F_i \delta t \quad \left(\begin{smallmatrix} i, & k=1, & \cdots, & m \\ l+k, & j+k, & \cdots, & i+j \end{smallmatrix}\right).$$

For example,

$$\begin{aligned} &A_{ik}[1, 2, \cdots, k-1, k+1, \cdots, m] \\ &= [1, 2, \cdots, k-1, k+1, \cdots, l-1, k, l+1, \cdots, m] \\ &= (-1)^{l-k-1}[1, 2, \cdots, k-1, k, k+1, \cdots, l-1, l+1, \cdots, m] \\ &= (-1)^{l+k-1} F_i. \end{aligned}$$

* Muir, Theory of Determinants, §§159, 163.

The transformation A_{ii} therefore gives the following increments:

$$\delta F_i = 0, \quad \delta F_k = F_k \delta t \quad (k = 1, \cdots, m;\ k \neq i).$$

The transformation $A_{ik}(i \neq k)$ produces the increments

$$\delta F_k = (-1)^{i+k-1} F_i \delta t, \quad \delta F_j = 0 \quad (j = 1, \cdots, m;\ j \neq k).$$

It is readily seen that the m^2 linearly independent infinitesimal transformations in the m variables F_k,

$$(6) \quad A_{ii} = \sum_{k \neq i}^{k=1, \cdots, m} F_k \frac{\partial f}{\partial F_k} \delta t; \quad A_{ik} = (-1)^{i+k-1} F_i \frac{\partial f}{\partial F_k} \delta t,$$

generate a group whose finite transformations are:

$$(7) \qquad F_i' = \sum_{j=1}^{m} A_{ij} F_j \qquad (j = 1, \cdots, m)$$

where A_{ij} is the minor (without sign) complementary to a_{ij} in the determinant $|a_{ij}|$. Indeed if we apply formula (5) to the determinant

$$A_{ij} \equiv \begin{vmatrix} 1 \cdots j-1 & j & j+1 \cdots i-1 & i+1 \cdots m \\ 1 \cdots j-1 & j+1 & j+2 \cdots i & i+1 \cdots m \end{vmatrix}$$

we find as in §6, the results

$$(8) \qquad A_{ii} = 1 + \sum_{s \neq i}^{s=1, \cdots, m} a_{ss} \delta t, \ A_{ij} = (-1)^{i+j-1} a_{ji} \delta t$$
$$(i, j = 1, \cdots, m;\ i \neq j).$$

It follows that the general infinitesimal transformation of the form (7) gives the following increments:

$$\delta F_i = \left[\sum_{s \neq i}^{s=1, \cdots, m} a_{ss} F_i + \sum_{j \neq i}^{j=1, \cdots, m} (-1)^{i+j-1} a_{ji} F_j \right] \delta t$$
$$(i = 1, \cdots, m),$$

from which we readily obtain the m^2 linearly independent transformations (6). We may therefore enunciate the following theorem* concerning the individual finite transformations of the above groups.

* This theorem is capable of proof by determinants without having recourse to the infinitesimal transformations of the groups concerned.

For m odd, the second compound $C_{m,2}$ of the general m-ary linear homogeneous group G_m possesses a system of m invariant Pfaffians,

$$F_i \equiv [1, 2, \cdots, i-1, i+1, \cdots, m] \quad (i = 1, \cdots, m).$$

The transformation $[a]_2$ of $C_{m,2}$, corresponding to any given transformation (a_{ij}) of G_m, effects upon the F_i a linear transformation which is identical with that m-ary transformation $[a]_{m-1}$ of the $(m-1)^{st}$ compound of G which corresponds to (a_{ij}).

10. For m even and $q = 2$, the skew-symmetric determinant $D_a^{(m-1)}$ vanishes identically. We readily find* that the bordered skew-symmetric determinants

$$D_a^{(m-1)} = [l, i_1, i_2, \cdots, i_{m-2}, k]\,[i_1, i_2, \cdots, i_{m-2}]$$
$$(l, k = 1, \cdots, m;\ l \neq k)$$

if $i_1, i_2, \cdots, i_{m-2}, l, k$ form a permutation of $1, 2, \cdots, m$.

It is readily verified that the transformations A_{ii} leave unaltered the Pfaffian $[1, 2, \cdots, m]$, while the A_{ij} $(i \neq j)$ annul it. Hence $[1, 2, \cdots, m]$ is an invariant of $C_{m,2}$. Consider the $\frac{1}{2}m(m-1)$ Pfaffians

$$F_{i_1 i_2 \cdots i_{m-2}} \equiv [i_1, i_2, \cdots, i_{m-2}].$$

We find that the transformation A_{ik} gives the increments,

$$\delta F_{i_1 i_2 \cdots i_{m-2}} = 0 \qquad \text{(if every } i_s \neq l);$$

$$\delta F_{i i_2 \cdots i_{m-2}} = F_{k i_2 \cdots i_{m-2}}\,\delta t.$$

But these are the increments produced by the transformation A_{ik} of the group $C_{m,m-2}$ upon its variables $F_{i_1 i_2 \cdots i_{m-2}}$ [see § 7]. We have therefore proved the following theorem, capable of proof using only the finite transformations of the groups involved:

For m even, the secono compound $C_{m,2}$ of the general m-ary linear group G_m possesses as an isolated invariant the Pfaffian $[1, 2, \cdots, m]$ and as a system of invariants the set of $C_{m,2}$ Pfaffians.

$$[i_1, i_2, \cdots, i_{m-2}] \qquad \left(\begin{smallmatrix} i_1, i_2, \cdots, i_{m-2} = 1, \cdots, m \\ i_1 < i_2 < \cdots < i_{m-2} \end{smallmatrix}\right).$$

The transformation $[a]_2$ of $C_{m,2}$, corresponding to any given transformation (a_{ij}) of G_m, effects upon these Pfaffians a linear transformation identical with that $C_{m,m-2}$ -ary transformation $[a]_{m-2}$ of the $(m-2)^{nd}$ compound of G_m, which corresponds to the given (a_{ij}).

* Muir, Theory of Determinants, § 163.

RECIPROCITY BETWEEN THE qTH AND THE $m-q$TH COMPOUNDS OF G_m, §§ 11–15.

11. We may express* the qth minors of the determinant $|A_{ij}|$ adjungate to $|a_{ij}|$ in terms of the $(m-q)$th minors of $|a_{ij}|$:

$$(9)\qquad \begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{vmatrix}_A = D^{q-1} \begin{Vmatrix} 1 & 2 & \cdots & i_1-1 & i_1+1 & \cdots & i_q-1 & i_q+1 & \cdots & m \\ 1 & 2 & \cdots & j_1-1 & j_1+1 & \cdots & j_q-1 & j_q+1 & \cdots & m \end{Vmatrix}_a,$$

the double bars indicating that, in the two series of integers written in ascending order, j_1-1 does not necessarily fall under i_1-1, etc.

If therefore we write, for every $i_1 < i_2 < \cdots < i_q \leqq m$,

$$Y_{1\,2\,\cdots\,i_1-1\,i_1+1\,\cdots\,i_q-1\,i_q+1\,\cdots\,m} \equiv Z_{i_1\,i_2\,\cdots\,i_q}$$

the general substitution $[a]_{m-q}$ of the group $C_{m,m-q}$ becomes

$$(10)\qquad Z'_{i_1 i_2 \cdots i_q} = D^{1-q} \sum \begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{vmatrix}_A Z_{j_1 j_2 \cdots j_q},$$

the summation extending over every combination $j_1, j_2, \cdots, j_q$ of the integers $1, \cdots, m$ taken q together.

12. To obtain the general infinitesimal transformation (10) we proceed as in § 6, using formulæ (8). We find

$$\begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ i_1 & i_2 & \cdots & i_q \end{vmatrix}_A = 1 + (q \sum_{s=1}^{m} a_{ss} - \sum_{s=1}^{q} a_{i_s i_s}) \delta t;$$

$$\begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{vmatrix}_A = 0 \qquad \text{(if two or more } j\text{'s differ from every } i\text{)};$$

$$\begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{vmatrix}_A = (-1)^{r+s-1} A_{i_r j_s} = (-1)^{r+s+r+s-1} a_{j_s i_r} \delta t,$$

if j_s be the only j different from every i, and i_r be the only i different from every j. Further,

$$D^{-q+1} = 1 + (-q+1) \sum_{s=1}^{m} a_{ss} \delta t.$$

* Compare Muir, end of § 97.

Hence $\delta Z_{i_1 i_2 \cdots i_q}$ equals δt times the expression

$$\Big(\sum_{s=1}^{m} a_{ss} - \sum_{s=1}^{q} a_{i_s i_s}\Big) Z_{i_1 i_2 \cdots i_q}$$
$$+ \sum_{r,s}^{1,\cdots,q} (-1)^{i_r + j_s + r + s - 1} a_{j_s i_r} Z_{i_1 \cdots i_{s-1} j_s i_{s+1} \cdots i_{r-1} i_{r+1} \cdots i_q}$$

summed also for $j_s = i_{s-1} + 1, \cdots, i_s - 1$. If we perform the summation with respect to s in the latter sum (see end of § 6), it becomes

$$\sum_{r,j} (-1)^{i_r + j + r} a_{j i_r} Z_{j i_1 \cdots i_{r-1} i_{r+1} \cdots i_q},$$

summed for

$$r = 1, \cdots, q; \quad j = 1, \cdots, m, \quad j + i_1, i_2, \cdots, \text{or } i_q.$$

13. Setting $a_{lk} = 1$ and the other a's equal zero, we obtain m^2 linearly independent infinitesimal transformations A_{lk}'. Setting

$$Q = \frac{\partial f}{\partial Z} \delta t,$$

and proceeding as in § 7, we find

$$A_{lk}' = (-1)^{i+k-1} \sum_{i_1, \cdots, i_q}^{1 \cdots m} Z_{li_1 \cdots i_{r-1} i_{r+1} \cdots i_q} Q_{ki_1 \cdots i_{r-1} i_{r+1} \cdots i_q}$$

$$(i_1 < i_2 \cdots < i_q, \text{ and each } i + l, + k).$$

$$A' - A_{kk}' = \sum_{i_1 \cdots i_q}^{1 \cdots m} Z_{ki_1 \cdots i_{r-1} i_{r+1} \cdots i_q} Q_{ki_1 \cdots i_{r-1} i_{r+1} \cdots i_q}$$

$$(i_1 < i_2 \cdots < i_q, \text{ and each } i + k),$$

where we denote by A' the following transformation

$$A' \equiv \sum_{i_1 \cdots i_q}^{1 \cdots m} Z_{i_1 i_2 \cdots i_q} Q_{i_1 i_2 \cdots i_q}$$

To prove that A' belongs to the group $C_{m, m-q}$ under consideration, we note that $A' - A_{kk}'$ contains $C_{m-1, q-1}$ terms, so that

$$mA' - \sum_{k=1}^{m} A_{kk}'$$

contains $mC_{m-1, q-1}$ terms which coincide in sets of q each, and among which every one of the $C_{m,q}$ terms of A' is represented. Hence, since $mC_{m-1, q-1} = qC_{m,q}$, it follows that

$$(m - q)A' = \sum_{k=1}^{m} A_{kk}'.$$

14. The set of m infinitesimal transformations of $C_{m,m-q}$,

$$A' - A_{lk}', \quad A_{ik}' \quad (l = 1, \cdots, m, l + k),$$

generate a group of m parameters which is identical with the group generated by the m transformations $A_{kl}(l=1, \cdots, m)$ of the group $C_{m,q}$. We thus see the exact manner in which the qth and $(m-q)$th compounds of the general m-ary linear group G_m are isomorphic.

When we confine ourselves to the group of those transformations of G_m of determinant $D=1$, the qth and the $(m-q)$th compounds are not merely isomorphic but identical. Indeed the m^2-1 transformations of the $C_{m,m-q}$,

$$A_{ik}' \ (l, k = 1, \cdots, m, l + k), \ A_{11}' - A_{kk}' \qquad (k = 2, \cdots, m)$$

are identical which the m^2-1 transformations

$$A_{kl} \ (k, l = 1, \cdots, m, k + l), \ A_{11} - A_{kk} \qquad (k = 2, \cdots, m)$$

of $C_{m,q}$, the corresponding transformations being given by the same pair of subscripts $(k\ l)$ or $(1\ k)$.

15. To illustrate the reciprocity between the groups $C_{m,q}$ and $C_{m,m-q}$ we take the example $m=5$, $q=2$. We write the table of §8 for the transformations of $C_{5,2}$ which belong to the set $l=2$; viz.,

	P_{21}	P_{23}	P_{24}	P_{25}
A_{23}	Y_{21}	Y_{23}	Y_{24}	Y_{25}
A_{21}	0	Y_{13}	Y_{14}	Y_{15}
A_{23}	$-Y_{13}$	0	Y_{34}	Y_{35}
A_{24}	$-Y_{14}$	$-Y_{34}$	0	Y_{45}
A_{25}	$-Y_{15}$	$-Y_{35}$	$-Y_{45}$	0

By §6 we obtain the following transformations of $C_{5,3}$:

	$-P_{345}$	P_{145}	P_{135}	P_{134}
$A' - A_{22}'$	$-Y_{345}$	Y_{145}	Y_{135}	Y_{134}
$+ A_{12}'$	0	Y_{245}	Y_{235}	Y_{234}
$+ A_{32}'$	$-Y_{245}$	0	Y_{125}	Y_{124}
$- A_{42}'$	$-Y_{235}$	$-Y_{125}$	0	Y_{123}
$+ A_{52}'$	$-Y_{234}$	$-Y_{124}$	$-Y_{123}$	0

We thus observe that any term as $Y_{245}P_{145}$ of the latter table may be derived at once from the corresponding term $Y_{13}P_{23}$ of the former by taking as subscripts to the one those

integers 1, 2, ···, 5 (in order), which do not occur among the subscripts to the other term. The rule which, if applied to the first table, gives the Pfaffian invariant $F_3 \equiv [1345]$ will, when applied to the second table, give

$$\overline{F}_3 \equiv -(Y_{23}Y_{45} - Y_{24}Y_{35} + Y_{25}Y_{34}),$$

which we will denote by $-_2[1345]$, the first subscript to the Y's being 2 throughout.

Forming the remaining four tables for the group $C_{5,2}$, and the corresponding tables for $C_{5,3}$, we obtain the following results:

$$F_1 \equiv [2345],\ \overline{F}_1 = {}_1[2345];\ F_2 \equiv [1245],\ \overline{F}_2 = {}_2[1245];$$
$$F_4 \equiv [1235],\ \overline{F}_4 = -{}_4[1235];\ F_5 \equiv [1234],\ \overline{F}_5 = {}_5[1234].$$

In general, if F_j or $F_{i_1 i_2 \cdots i_{m-2}}$ denote the Pfaffians formed from the tables of the transformations A_{jk} of $C_{m,2}$, we will denote by $\overline{F}_j$ or $\overline{F}_{i_1 i_2 \cdots i_{m-2}}$ the Pfaffians formed from the corresponding tables of the transformations A_{jk}', $A' - A_{jk}'$ of $C_{m,m-2}$.

GROUP INDUCED BY $C_{m,m-2}$ UPON ITS INVARIANTS, §§ 16–18.

16. For m odd and $q = 2$, the group $C_{m,m-q}$ has a system of m invariant Pfaffians $\overline{F}_j$ of degree $\frac{1}{2}(m-1)$. By §9, the transformation $A' - A_{jk}'$ effects upon the $\overline{F}_j$ the transformation

$$\sum_{j \neq k}^{j=1,\cdots,m} \overline{F}_j \frac{\partial f}{\partial \overline{F}_i} \delta t;$$

while $A_{jk}' \equiv (-1)^{k+i-1} A_{ki}$ produces the transformation

$$\overline{F}_k \frac{\partial f}{\partial \overline{F}_i} \delta t.$$

Since the Eulerian operator A' multiplies each $\overline{F}_j$ by $\frac{1}{2}(m-1)$, it follows that A_{jk}' produces the following transformation:

$$\tfrac{1}{2}(m-1)\sum_{j=1}^{m} \overline{F} \frac{\partial f}{\partial \overline{F}_j} \delta t - \sum_{j \neq k}^{j=1,\cdots,m} \overline{F}_j \frac{\partial f}{\partial \overline{F}_j} \delta t$$
$$\equiv \tfrac{1}{2}(m-3)\sum_{j \neq k}^{j=1,\cdots,m} \overline{F}_j \frac{\partial f}{\partial \overline{F}_j} \delta t + \tfrac{1}{2}(m-1)\ \overline{F}_k \frac{\partial f}{\partial \overline{F}_k} \delta t.$$

The finite transformations of the group induced upon the $\overline{F}_j$ by the group $C_{m,m-2}$ have therefore the form

$$(11) \qquad \overline{F}_i' = D^{\frac{m-3}{2}} \sum_{j=1}^{m} a_{ij} \overline{F}_k \qquad (i = 1, \cdots, m).$$

17. For m even and $q = 2$, the group $C_{m,m-q}$ has as an isolated invariant a Pfaffian of degree $m/2$ and as a system of invariants the C_m. Pfaffians $\overline{F}_{i_1 i_2 \cdots i_{m-2}}$ of degree $\frac{1}{2}(m-2)$. It follows from §§ 10, 13, 14 that the transformation $(-1)^{l+k-1} A_{lk}'$ $(l \neq k)$ of $C_{m,m-2}$ gives rise to the following increments in the Pfaffian invariants:

$$(a) \qquad \begin{cases} \delta \overline{F}_{i_1 i_2 \cdots i_{m-2}} = 0 & \text{(if each } i \neq k) \\ \delta \overline{F}_{k i_2 \cdots i_{m-2}} = \overline{F}_{l i_2 \cdots i_{m-2}} \delta t; \end{cases}$$

also that $A' - A_{kk}'$ produces the increments (a) (when l is replaced by k). Since A' multiplies each $\overline{F}_{i_1 \cdots i_{m-2}}$ by $\frac{1}{2}(m-2)$, it follows that A_{kk}' produces the increments

$$(b) \qquad \begin{cases} \delta \overline{F}_{i_1 i_2 \cdots i_{m-2}} = \frac{1}{2}(m-2) \overline{F}_{i_1 i_2 \cdots i_{m-2}} \delta t \quad \text{(if each } i \neq k), \\ \delta \overline{F}_{k i_2 \cdots i_{m-2}} = \frac{1}{2}(m-4) \overline{F}_{k i_2 \cdots i_{m-2}} \delta t. \end{cases}$$

Having thus determined the infinitesimal transformations of the group induced by the group $C_{m,m-2}$ upon its system of invariants $\overline{F}_{i_1 \cdots i_{m-2}}$, we may readily show that the finite transformations of this group are

$$(12) \qquad \overline{F}'_{i_1 i_2 \cdots i_{m-2}} = D^{1-\frac{m}{2}} \sum_{j_1, \cdots, j_{m-2}}^{1 \cdots m} \begin{vmatrix} i_1 \cdots i_{m-2} \\ j_1 \cdots j_{m-2} \end{vmatrix} A \overline{F}_{j_1 j_2 \cdots j_{m-2}}.$$

Indeed, proceeding as in §§ 11–13, we find that the infinitesimal transformation gotten from (12) by setting $a_{kk} = 1$ and the other a's $= 0$ has precisely the increments (b), while that given by setting $a_{ik} = 1$ and the other a's $= 0$ has, when multiplied by $(-1)^{i+k-1}$, precisely the increments (a).

To give (12) another form, we set

$$\overline{F}_{i_1 i_2 \cdots i_{m-2}} \equiv W_{i_{m-1} i_m} \qquad \left(\begin{matrix} i_2 < i_2 < \cdots < i_{m-2} \\ i_{m-1} < i_m \end{matrix} \right),$$

when i_{m-1} is the first and i_m the second integer $< m$ which does not occur in the series $i_1, i_2, \cdots, i_{m-2}$.

Further, formula (9) of §11 becomes for $q = m - 2$

$$\begin{vmatrix} i_1 i_2 \cdots i_{m-2} \\ j_1 i_2 \cdots j_{m-2} \end{vmatrix} A = D^{m-3} \begin{vmatrix} i_{m-1} i_m \\ j_{m-1} j_m \end{vmatrix} a.$$

Hence the transformation (12) takes the form*

$$(12_1) \qquad W'_{i_{m-1} i_m} = D^{\frac{m-4}{2}} \sum_{j_{m-1} j_m}^{1, \cdots, m} \begin{vmatrix} i_{m-1} i_m \\ j_{m-1} j_m \end{vmatrix} a W_{j_{m-1} j_m}.$$

18. We may enunciate the results proven in §§ 16–17 for the individual transformations of the groups concerned:

To any given transformation (a_{ij}) of determinant D of the general m-ary linear homogeneous group G_m, there corresponds a transformation $[a]_{m-2}$ of the $(m-2)^{d}$ compound $C_{m,m-2}$ which gives rise to a linear transformation upon its system of Pfaffian invariants, viz:

1°: *for m odd, the m-ary transformation,*

$$\overline{F_i'} = D^{\frac{m-3}{2}} \sum_{j=1}^{m} a_{ij} \overline{F_j} \qquad (i = 1, \cdots, m),$$

which for $D = 1$, is precisely the given transformation of G_m.

2°. *for m even, the $\frac{1}{2}m(m-1)$-ary transformation (12) or (12_1), where, for $D = 1$, (12_1) belongs to the second compound of G_m, and (12) to the $(m-2)^{d}$ compound of the $(m-1)^{st}$ compound of G_m.*

UNIVERSITY OF CALIFORNIA,
August 9, 1898.

A SECOND LOCUS CONNECTED WITH A SYSTEM OF COAXIAL CIRCLES.

BY PROFESSOR THOMAS F. HOLGATE.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, August 19, 1898.)

IN a paper read before this Society at its Toronto Meeting and published in the BULLETIN for November, 1897, I

* We may verify (12_1) directly, using the method of § 6 for $q = 2$. The presence of the factor $D^{\frac{m-4}{2}}$ influences only the transformations $A_{kk'}$. There occurs, however, some difficulty as to signs in passing from the W's to the F's. Likewise the results of §§ 11–14 could doubtless be proved by the method of § 6.

presented the locus of the common point of three concurrent lines which are tangent to three circles of a coaxial system, each line tangent to two of the three circles and each circle touched by two of the three lines, the circles and lines thus forming a continuous chain, which for a proper or non-degenerate circuit of the kind described proved to be a pair of parabolas having the line of centers of the coaxial system for a common directrix and the two finite points of intersection of the circles for foci.

For simplicity in performing the necessary eliminations in the former paper one of the chosen lines was drawn parallel to the line of centers, but if this first line be selected at random the eliminations do not prove so difficult as they at first appeared, and, the process being of a more general character, other interesting features of the problem come to the surface. In this paper I propose to present these features and then proceed to the case of period four, *i. e.*, to the case in which the circuit closes with four lines and four circles instead of three.

If now we choose the equation for any one of the circles of the coaxial system, as before, in the form

$$x^2 + y^2 - 2k_\nu x - \delta^2 = 0$$

and the equation of a line through the point (a, b) in the form

$$(x - a) + l_\nu(y - b) = 0,$$

where k_ν and l_ν are parameters determining the particular circle and line under consideration, we find the conditions of contact to be

$$(1)\quad k_1^2 l_1^2 + (\delta^2 - b^2) l_1^2 + 2bk_1 l_1 + 2ak_1 - 2abl_1 + (\delta^2 - a^2) = 0,$$
$$(2)\quad k_1^2 l_2^2 + (\delta^2 - b^2) l_2^2 + 2bk_1 l_2 + 2ak_1 - 2abl_2 + (\delta^2 - a^2) = 0,$$
$$(3)\quad k_2^2 l_2^2 + (\delta^2 - b^2) l_2^2 + 2bk_2 l_2 + 2ak_2 - 2abl_2 + (\delta^2 - a^2) = 0,$$
$$(4)\quad k_2^2 l_3^2 + (\delta^2 - b^2) l_3^2 + 2bk_2 l_3 + 2ak_2 - 2abl_3 + (\delta^2 - a^2) = 0,$$
$$(5)\quad k_3^2 l_3^2 + (\delta^2 - b^2) l_3^2 + 2bk_3 l_3 + 2ak_3 - 2abl_3 + (\delta^2 - a^2) = 0,$$
$$(6)\quad k_3^2 l_1^2 + (\delta^2 - b^2) l_1^2 + 2bk_3 l_1 + 2ak_3 - 2abl_1 + (\delta^2 - a^2) = 0,$$

a set of equations involving the coördinates (a, b) of the point of concurrence and the parameters l_1, l_2, l_3 of the three lines and k_1, k_2, k_3 of the three circles.

From these six equations it would clearly be possible to

eliminate a, b, l_1, l_2, l_3, or a, b, k_1, k_2, k_3, and thus obtain a relation among the k's, or among the l's, which would render the circuit possible. Proceeding, however, to find the locus of the point of concurrence for which the circuit is possible, we shall eliminate k_1 from equations (1) and (2). k_2 from equations (3) and (4), and k_3 from equations (5) and (6). These eliminations yield the following three expressions which show the two to two relations among the l's, the geometric significance of which will be clear from a diagram, viz.:

$$(A)\quad pl_1^2l_2^2 + ql_1l_2(l_1 + l_2) + r(l_1^2 + l_2^2) + sl_1l_2 + t(l_1 + l_2) = 0,$$

$$(B)\quad pl_2^2l_3^2 + ql_2l_3(l_2 + l_3) + r(l_2^2 + l_3^2) + sl_2l_3 + t(l_2 + l_3) = 0,$$

$$(C)\quad pl_3^2l_1^2 + ql_3l_1(l_3 + l_1) + r(l_3^2 + l_1^2) + sl_3l_1 + t(l_3 + l_1) = 0,$$

where p is written instead of $4b^2(\delta^2 + a^2 - b^2)$,

q, instead of $4ab(\delta^2 + a^2 - 2b^2)$,

r, instead of $(\delta^4 + 2\delta^2a^2 + a^4 - 4a^2b^2)$,

s, instead of $2(\delta^4 + 2\delta^2a^2 + a^4 - 2\delta^2b^2 - 6a^2b^2)$,

t, instead of $-4ba(\delta^2 + a^2)$.

If l_1 be given any fixed value, equation (A) yields the same two roots for l_2 as equation (C) yields for l_3, hence one of these roots must be the value of l_3, the other the value of l_2, consistent with the chosen value of l_1, as is geometrically evident; so that

$$l_2 + l_3 = -\frac{ql_1^2 + sl_1 + t}{pl_1^2 + ql_1 + r},$$

$$l_2l_3 = \frac{rl_1^2 + tl_1}{pl_1^2 + ql_1 + r}.$$

Substituting these expressions in equation (B) and simplifying we obtain the relation

$$(P)\quad (rs - qt - r^2)\,[pl_1^4 + 2ql_1^3 + (2r + s)l_1^2 + 2tl_1] = 0.$$

If now the coördinates of the point of concurrence satisfy the relation

$$rs - qt - r^2 = 0,$$

then no matter how the line l_1 is drawn, the proposed system is possible.

Replacing q, r, s, t in this last expression by the functions

of a and b which they symbolize, and writing x and y for variables, we obtain for the locus of the point of concurrence the equation

$$(\delta^4 + 2\delta^2x^2 + x^4 + 4x^2y^2)\,(\delta^2 + x^2 - 2\delta y)\,(\delta^2 + x^2 + 2\delta y) = 0.$$

The curve represented by the first factor of this equation is wholly imaginary, while those represented by the second and third factors are parabolas, each having the line of centers of the circles for directrix and a finite point of intersection of the circles for focus as was deduced in the former paper. If the circles of the coaxial system have imaginary intersections, *i. e.*, if δ^2 be negative and hence δ imaginary, the locus becomes wholly imaginary, except in two points, viz., the two limiting points of the system of circles. To make the geometric necessity for this evident we need only to recall that of three circles of such a system one must lie wholly within another.

The conditions necessary for a complete circuit will however be satisfied for any point (a, b) of the plane if some one of the three tangent lines be so chosen as to make the second factor in equation (P) vanish, that is, so as to make

$$pl_1^4 + 2ql_1^3 + (2r + s)l^2 + 2tl = 0.$$

In this case the parameter l of the line must equal zero, or

$$\frac{-a \pm i\delta}{b}, \text{ or } \frac{2ab}{\delta^2 + a^2 - b^2},$$

the first three of these values locating the line l through one or other of the three vertices of the common self-conjugate triangle of the coaxial system, the fourth locating it as the tangent to that circle of the system which passes through the point a, b.

This second factor is in fact simply the form which equation (A), or (B), or (C) would take if the two l's involved were equal, and so if l_2, say, should assume any of the three values which would direct the corresponding line through a vertex of the self-conjugate triangle, as for example, the value zero, then in equation (B) one value of l_2 would also be zero while the other, *i. e.*, l_1, would be of the form

$$\frac{4ab(\delta^2 + a^2)}{\delta^4 + 2\delta^2a^2 + a^4 - 4a^2b^2}.$$

These values for the three l's would give for the k's the values

$$k_2 = \infty, \quad k_1 = k_3 = \frac{a^2 - \delta^2}{2a}.$$

If on the other hand, l_3, should assume the value

$$\frac{2ab}{\delta^2 + a^2 - b^2}$$

which locates the line as tangent to that circle of the system passing through a, b, then l_2 would equal l_3, and l_1 would assume a definite value different from them. The corresponding values for the k's would be

$$k_1 = k_3 = -\frac{\delta^4 + 2\delta^2 a^2 + a^4 - \delta^2 b^2 + a^2 b^2}{2ab^2}, \quad k_2 = \frac{\delta^2 - a^2 - b^2}{-2a}.$$

Hence in either of these cases we obtain a degenerate or improper circuit of period three in which two of the lines involved coincide as do also two of the circles, the lines and circles following each other in the order indicated by the succession of parameters $l_1, k_1, l_2, k_2, l_3, k_3$, returning to the line whose parameter is l_1, and so closing the circuit.

Thus for any point of the plane as point of concurrence there are four distinct circuits of period three while for any point of our locus we have not only these four improper circuits, but also an infinite number of proper circuits.

Turning now to the case in which the circuit closes with four lines and four circles, we have, instead of the six equations of contact, the following eight, viz.:

$$f(k_1, l_1) = 0, \quad f(k_1, l_2) = 0, \quad f(k_2, l_2) = 0, \quad f(k_2, l_3) = 0,$$
$$f(k_3, l_3) = 0, \quad f(k_3, l_4) = 0, \quad f(k_4, l_4) = 0, \quad f(k_4, l_1) = 0,$$

where $f(k_\mu, l_\nu)$ is as before of the form

$$k_\mu^2 l_\nu^2 + (\delta^2 - b^2) l_\nu^2 + 2bk_\mu l_\nu + 2ak_\mu - 2abl_\nu + (\delta^2 - a^2).$$

Using the same abbreviations as for period three and eliminating the k's from the eight conditions of contact, two and two, we obtain

$$(A)\ pl_1^2 l_2^2 + ql_1 l_2 (l_1 + l_2) + r(l_1^2 + l_2^2) + sl_1 l_2 + t(l_1 + l_2) = 0,$$
$$(B)\ pl_2^2 l_3^2 + ql_2 l_3 (l_2 + l_3) + r(l_2^2 + l_3^2) + sl_2 l_3 + t(l_2 + l_3) = 0,$$
$$(C)\ pl_3^2 l_4^2 + ql_3 l_4 (l_3 + l_4) + r(l_3^2 + l_4^2) + sl_3 l_4 + t(l_3 + l_4) = 0,$$
$$(D)\ pl_4^2 l_1^2 + ql_4 l_1 (l_4 + l_1) + r(l_4^2 + l_1^2) + sl_4 l_1 + t(l_4 + l_1) = 0,$$

If in equation (A) we should give l_1 any particular value and solve for l_2 we should obtain the same values as would be obtained for l_4 from equation (D) when the same value is assumed for l_1. But these are merely the values of l_2 or l_4 for which lines are tangent to the same circles of the coaxial system as is the line l_1, that is, one of these roots is l_2 and the other is l_4.

Hence

$$l_2 + l_4 = -\frac{ql_1^2 + sl_1 + t}{pl_1^2 + ql_1 + r},$$

$$l_2 l_4 = \frac{rl_1^2 + tl_1}{pl_1^2 + ql_1 + r}.$$

Similar considerations will yield from equations (B) and (C) the relations

$$l_2 + l_4 = -\frac{ql_3^2 + sl_3 + t}{pl_3^2 + ql_3 + r},$$

$$l_2 l_4 = \frac{rl_3^2 + tl_3}{pl_3^2 + ql_3 + r}.$$

Equating these two values for $l_2 + l_4$ and for $l_2 l_4$ we obtain, in case $l_1 \neq l_3$, the two relations

$$(Q) \quad (q^2 - ps)l_1 l_3 + (qr - pt)(l_1 + l_3) + (rs - qt) = 0$$

$$(R) \quad (qr - pt)l_1 l_3 + r^2(l_1 + l_3) + rt = 0.$$

If the equations (A), (B), (C), (D) had been taken in pairs (A), (B) and (C), (D), and given a similar treatment we should have obtained relations between l_2 and l_4 identical with those between l_1 and l_3. In fact, it is easily seen that either pair of lines l_1, l_3, or l_2, l_4, is sufficient to determine the whole circuit, since either pair determines all four circles of the circuit, the circles touched by one line of the pair being different from those touched by the other line of the pair.

Since equations (Q) and (R) are to be satisfied for all values of l_1 and all consistent values of l_3, then must

$$\frac{q^2 - ps}{qr - pt} = \frac{qr - pt}{r^2} = \frac{rs - qt}{rt};$$

and these relations are satisfied in case

$$pt^2 - 2qrt + r^2 s = 0,$$

which gives us the desired locus.

Replacing p, q, r, s, t again by the functions of a and b which they represent and writing x and y for variables we have for the equation of the proper circuit of period four

$$(\delta^8 + 4\delta^6x^2 + 6\delta^4x^4 + 4\delta^2x^6 + x^8 + 16\delta^3x^2y^4)(\delta^4 + 2\delta^2x^2 + x^4 - 2\delta^2y^2 + 2x^2y^2) = 0.$$

The first factor of this locus represents a curve which is wholly imaginary when the circles of the coaxial system have real intersections, *i. e.*, when δ^2 is positive, and which breaks up into two imaginary branches, each of which has real double points at the limiting points of the system when δ^2 is negative.

In case the coaxial system has real intersections, the second factor of the locus represents a real curve of the fourth order which is symmetrically situated with respect both to the line of centers and to the radical axis. It cuts the line of centers only in imaginary points and crosses the radical axis orthogonally at the points $0, \pm \frac{\delta}{\sqrt{2}}$. The curve has an ordinary double point at infinity on the radical axis, the asymptotes being $x = \pm\delta$, between which lines the curve is wholly confined, and two imaginary double points at $\pm i\delta$, 0. If the coaxial system has imaginary intersections, the curve represented by this factor becomes also imaginary except for two real double points at the limiting points of the system.

If in equation (A) we impose the condition that the values of l_2 for any given l_1 shall be equal, *i. e.*, that l_2 shall equal l_4, and hence that the circuit shall degenerate, we find that

$$(q^2 - 4pr)l_1^4 + 2(qs - 2pt - 2qr)l_1^3 + (s^2 - 2qt - 4r^2)l_1^2 + 2(st - 2rt)l_1 + t^2 = 0,$$

or substituting the proper values for p, q, r, s, t, this becomes

$$16b^2(\delta^2 + a^2)[(\delta - b)l_1 - a][(\delta + b)l_1 + a][l_1 + i][l_1 - i] = 0.$$

Hence if the point of concurrence of the lines of the circuit be anywhere on the line $y = 0$ or on either of the lines $\delta^2 + x^2 = 0$, or if for any point of the plane, l_1 be so chosen as to equal

$$\frac{a}{\delta - b'} \quad \frac{-a}{\delta + b'} - i, \text{ or } i,$$

that is to say, if for any point of the plane one of the lines of the circuit passes through either of the finite intersection points of the circles or through either of the circular points at infinity (through any of the four common points of the circles of the system) the circuit will degenerate.

It is readily seen from an examination of the geometrical figure that such a degenerate circuit is possible for any point of the plane. If from any point P a straight line be drawn to one of the points of intersection of the coaxial system, the two circles of the system tangent to this line coincide. The second tangent through P to this circle determines a second circle which in turn determines a third line, and so on. If now we start with this last line, designating it l_1, and retrace our steps through line and circle alternately, $l_1, k_1, l_2, k_2, l_3, k_3, l_4, k_4$, we shall find that our circuit closes, or that we return to the starting line l_1, with an even number of lines and of circles, also that in the case of period four l_3 coincides with l_1 while l_1 and l_2 are distinct and that k_1 coincides with k_4 and k_2 with k_3. It will of course be remembered that a cyclic permutation of parameters in 'no way affects the circuit and that for period four the parameters whose subscripts are *one* and *three* or *two* and *four* are interchangeable as are also the pairs *one, three* and *two, four*.

The whole matter of degenerate or improper circuits may be summarized as follows:

If for any point of the plane, and starting with any line through it, we traverse a chain of lines and circles of the kind under consideration, and in our progress come upon a line passing through one of the intersection points of the system of circles, then the chain will return upon itself and will close with an even number of lines and of circles. If we recall the treatment of the degenerate circuits of order three it will be readily seen that:

Starting with any line, if we traverse a chain of the kind under consideration and in our progress come upon a line passing through a vertex of the self-conjugate triangle of the system of circles or upon the tangent to the circle through the point of concurrence, then the chain will return upon itself and will close with an odd number of lines and of circles.

For any point of the plane then there will always be four (degenerate) circuits of any odd period and four of an even

period. If for a point there should be a fifth circuit of any period then there will be an infinite number of circuits of the same period. That such points exist for any particular period appears at once from a study of the number of conditions of contact and the number of parameters involved in them. The process here employed is adequate to produce the locus of points which admit circuits of any period, but for periods higher than four the eliminations become exceedingly complicated.

Many interesting phases of this problem appear by making certain transformations of the plane. For instance, a projection of the plane will convert our configuration into the more general one consisting of lines through a point and an equal number of conics through four points, each line tangent to two conics, and each conic touched by two lines. An inversion would convert the lines of the circuit into circles of a coaxial system, leaving the circles of the circuit still circles of a coaxial system. Thus our configuration would come to consist of a given number of circles of one coaxial system and an equal number of a second coaxial system, each circle of either set touching two of the other set, the whole forming a continuous chain. Our locus would then become the locus of one of the four intersection points of the two systems of circles, which moves, the other three remaining fixed, so as always to make such a chain of given period possible.

NORTHWESTERN UNIVERSITY,
July, 1898.

RECIPROCAL TRANSFORMATIONS OF PROJECTIVE COÖRDINATES AND THE THEOREMS OF CEVA AND MENELAOS.

BY PROFESSOR ARNOLD EMCH.

1. Among the great number of correspondences between certain configurations of the plane and space it is interesting and valuable to consider relations of the triangle in connection with certain surfaces. It will be seen that propositions of plane geometry interpreted in Cartesian space lead to geometrical questions of a more general character In this paper we shall confine ourselves to the theorems of

Ceva and *Menelaos* and their connection with certain transformations of plane and space.

I. *On the theorem of Ceva.*

2. The projective coördinates* x_1, x_2, x_3 of a point P with regard to a triangle $A_1A_2A_3$ and its center of gravity E, (Fig. 1), as the unit point are defined by the ratios

$$\frac{A_3P_1}{A_2P_1} = \frac{x_2}{x_3}, \quad \frac{A_1P_2}{A_3P_2} = \frac{x_3}{x_1}, \quad \frac{A_2P_3}{A_1P_3} = \frac{x_1}{x_2}. \tag{1}$$

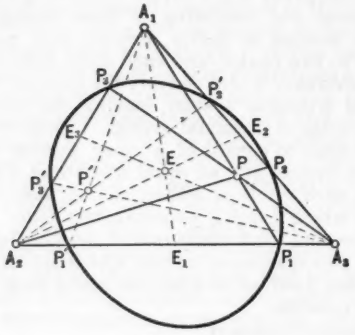

FIG. 1.

By the reciprocal transformation

$$x_1' : x_2' : x_3' = \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}$$

the point P is transformed into a point P' so that its coördinates are expressed by the ratios

$$\frac{x_2'}{x_3'} = \frac{x_3}{x_2}, \quad \frac{x_3'}{x_1'} = \frac{x_1}{x_3}, \quad \frac{x_1'}{x_2'} = \frac{x_2}{x_1}. \tag{2}$$

As this transformation is well known it will be sufficient for our purpose to state some of its most important properties.

A straight line

$$a_1x_1' + a_2x_2' + a_3x_3' = 0 \tag{3}$$

* Fiedler, Geometrie der Lage, vol. 3, p. 74.

is transformed into the conic

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 = 0 \qquad (4)$$

which passes through the vertices of the triangle $A_1A_2A_3$. To the line at infinity corresponds a conic K, and it is easily seen that the conics corresponding to straight lines are hyperbolas, parabolas, or ellipses, according as these straight lines respectively intersect, touch, or do not intersect the conic K. To every straight line through a vertex corresponds a straight line through the same vertex. The six rays

$$x_1 \pm x_2 = 0, \quad x_2 \pm x_3 = 0, \quad x_3 \pm x_1 = 0, \qquad (5)$$

including the medians of the triangle, and their four points of intersection, including the center of gravity, form the invariant elements of the reciprocal transformation. In general a curve of the nth order is transformed into a curve of the $2n$th order. If a curve of the nth order passes through the vertices A_1, A_2, A_3, the corresponding curve consists of a curve of the order $2n - 3$ and the three sides of the fundamental triangle. Thus a curve of the third order

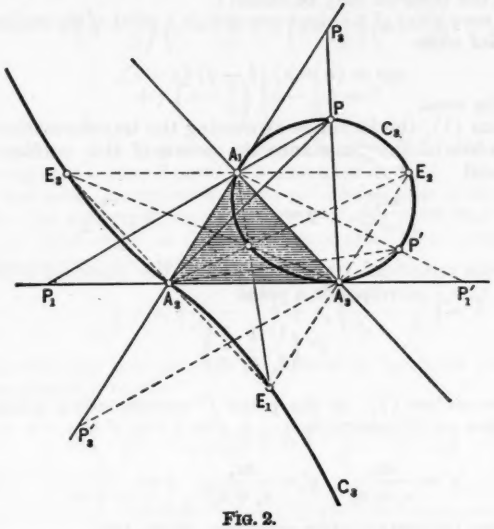

FIG. 2.

passing through the points A_1, A_2, A_3 is transformed into a curve of the third order through the same points. From the equation

$$a_1x_1(x_2^2 - x_3^2) + a_2x_2(x_3^2 - x_1^2) + a_3x_3(x_1'^2 - x_2^2) = 0 \quad (6)$$

it is seen that all curves of the third order passing through the vertices A_1, A_2, A_3 and the invariant points E, E_1, E_2, E_3 (Fig. 2) are invariant. Every point P of the curve is transformed into a point P' of the same curve.

3. The relation between the plane of the triangle $A_1A_2A_3$ and space is now established by introducing the new variables $A_2P_1 = x$, $A_3P_2 = y$, $A_1P_3 = z$, and $A_3P_1 = a - x$, $A_1P_2 = b - y$, $A_2P_3 = c - z$. Considering x, y, z as the Cartesian coördinates of a point Q in space, it is evident that to every point P of the plane with the projective coördinates x_1, x_2, x_3 corresponds a point Q in space with the coördinates x, y, z. But according to the *theorem of Ceva*,* these quantities are subject to the condition (fig. 1)

$$A_2P_1 \cdot A_3P_2 \cdot A_1P_3 = A_3P_1 \cdot A_1P_2 \cdot A_2P_3.$$

Thus the theorem may be stated:

To every point of the plane corresponds a point of the surface of the third order

$$xyz = (a - x)(b - y)(c - z), \quad (7)$$

and vice versa.

From (1), the formulas expressing the transformation of the points of the plane into the points of this surface are obtained

$$x = \frac{ax_3}{x_3 + x_2}, \quad y = \frac{bx_1}{x_1 + x_3}, \quad z = \frac{cx_2}{x_2 + x_1}. \quad (8)$$

In the reciprocal transformation of the plane, to a point $P(x_1, x_2, x_3)$ corresponds a point

$$P'\left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right).$$

On the surface (7), to the point P' corresponds a point Q' with the coördinates $x' = a - x$, $y' = b - y$, $z' = c - z$, or

$$x' = \frac{ax_3}{x_3 + x_2}, \quad y' = \frac{bx_3}{x_1 + x_3}, \quad z' = \frac{cx_1}{x_2 + x_1}. \quad (9)$$

* Ceva, De lineis se invicem secantibus. Milan, 1678.

Considering the coördinates of the points Q and Q', it is easily seen that the surface is symmetrical with regard to a center S which lies on the surface and whose coördinates are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$. We have, therefore, the theorem:

The reciprocal transformation of the projective coördinates of the plane may be interpreted as a symmetrical transformation of a surface of the third order with regard to a center which lies on the surface. The surface itself is defined by the analytical expression of the theorem of Ceva concerning the transversals of a triangle.

4. The character of this transformation and the surface appears from a few simple facts. In the reciprocal transformation of the plane, to a straight line g corresponds a conic g'. In the transformations (8) and (9) of the plane into the surface, the lines g and g' are transformed into two curves of the third order on the surface. These curves evidently are symmetrical with regard to the center S and lie therefore on a conical surface whose vertex coincides with S. As the intersection of this cone and the surface consists of two curves of the third order it necessarily follows that the cone is of the second order.

Its equation has the form

$$\alpha\left(y-\frac{b}{2}\right)\left(z-\frac{c}{2}\right)+\beta\left(z-\frac{c}{2}\right)\left(x-\frac{a}{2}\right)+\gamma\left(x-\frac{a}{2}\right)\left(y-\frac{b}{2}\right)=0 \tag{10}$$

and shows that the cone passes through the lines drawn through the center S parallel to the axes X, Y, Z. *There are* ∞^2 *such cones, corresponding to the* ∞^2 *straight lines of the plane.*

In the reciprocal transformation of the plane the curve of the third order as defined by equation (6) is invariant. Its equation may also be written

$$a_1\left(\frac{x_2}{x_3}-\frac{x_3}{x_2}\right)+a_2\left(\frac{x_3}{x_1}-\frac{x_1}{x_3}\right)+a_3\left(\frac{x_1}{x_2}-\frac{x_2}{x_1}\right)=0. \tag{11}$$

Transforming this equation by means of formulas (1), the new equation results

$$a_1\left(\frac{a-x}{x}-\frac{x}{a-x}\right)+a_2\left(\frac{b-y}{y}-\frac{y}{b-y}\right)+a_3\left(\frac{c-z}{z}-\frac{z}{c-z}\right)=0. \tag{12}$$

It is the equation of a cone which intersects the surface in a curve that corresponds to the invariant curve of the third order in the plane. Excluding the singular elements due to the vertices of the fundamental triangle, this cone reduces to a plane whose equation is easily found as

$$\frac{x}{a}(a_2+a_3)+\frac{y}{b}(a_3+a_1)+\frac{z}{c}(a_1+a_2) - (a_1+a_2+a_3)=0. \tag{13}$$

It passes through the center of the surface, and as there are ∞^2 such planes the theorem follows:

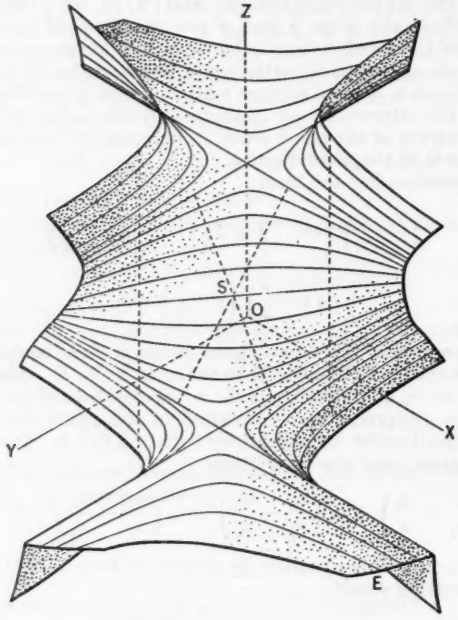

FIG. 3.

To the ∞^2 invariant curves of the third order passing through the vertices and the invariant points of the fundamental triangle in the plane correspond one by one the ∞^2 curves of the third order

formed by the intersection of all planes passing through the center S of the surface (7).

Without entering into further details we shall confine ourselves to giving a picture of the surface in question (Fig. 3). It is obtained by tracing the hyperbolas which correspond to the straight lines in the plane passing through the vertex A_3 of the fundamental triangle.

II. *On the Theorem of Menelaos.*

5. Assuming again a fundamental triangle $A_1A_2A_3$ we can also consider the straight line as the element of the plane. Let the straight line p intersect the sides of the fundamental triangle in the points P_1, P_2, P_3 and put $A_2P_1 = x$, $A_3P_2 = y$, $A_1P_3 = z$, $A_1A_2 = a$, $A_2A_3 = b$, $A_3A_1 = c$, then, according to the theorem of Menelaos,*

$$xyz = -(a-x)(b-y)(c-z). \tag{14}$$

If x, y, z are again considered as the coördinates of a point Q in space, it is clear that to the ∞^2 straight lines of the plane correspond ∞^2 points in space which form a surface with (14) as its equation.

This equation may also be written in the form

$$ayz + bzx + cxy - bcx - cay - abz + abc = 0, \tag{15}$$

and represents a hyperboloid of one sheet. If (x, y, z) is a point of the surface, the same is true of the point $(a-x, b-y, c-z)$. Thus, as in the previous case, the point S with the coördinates $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ is a center of symmetry of the surface, viz., the center of the hyperboloid. In the plane of the fundamental triangle this fact is expressed by the reciprocal transformation of the projective coördinates of the straight line

$$\xi_1' : \xi_2' : \xi_3' = \frac{1}{\xi_1} : \frac{1}{\xi_2} : \frac{1}{\xi_3}. \tag{16}$$

By this transformation, to a point of the plane corresponds a conic which touches the sides of the fundamental triangle. The line at infinity is invariant. *Thus, to a point at infinity corresponds a parabola which is inscribed in the fundamental triangle.*

The transformation of the plane into the hyperboloid is

* Found by Menelaos about 100 A. D.

made in a similar way as in the interpretation of the theorem of Ceva. It is easily found that *to a point and its corresponding conic in the plane correspond two curves of the third order on the hyperboloid which are symmetrical with regard to the center S.*

To the infinite points of the plane (pencils of parallel rays) and the parabolas inscribed to the fundamental triangle correspond the rectilinear elements of the hyperboloid.

6. From the above statement it is evident that the properties concerning the interpretation of the theorems of Ceva and Menelaos might be multiplied. In a similar manner we might interpret other relations of the triangle and thus obtain a special kind of surfaces whose properties are connected with the triangle. Such surfaces have been found by Rosace* and Steiner. Considered from this point of view, the investigations of Tucker, Taylor, McCay, Lemoine, Brocard, and others in the so-called "modern geometry of the triangle" would probably also lead to a number of valuable facts.†

BIEL, SWITZERLAND,
June, 1898.

NOTES.

A NEW list of members of the AMERICAN MATHEMATICAL SOCIETY will be published in January. Blank forms for furnishing necessary information have been sent to each member, and a full and prompt response is requested.

THE annual meeting of the London Mathematical Society was held on the evening of November 10th. LORD KELVIN consented to be nominated for the presidency, and Professors E. B. ELLIOTT and H. LAMB and Lieut.-Col. A. J. C. CUNNINGHAM for the vice-presidencies. The retiring members of the Council are Messrs. M. JENKINS and G. B. MATHEWS.

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* Rosace, p. 170 of *L'intermédiaire des mathématiciens*, No. 8, vol. 4. The equation of this reference is

$$R^2(x+y+z)(x+y-z)(x-y+z)(-x+y+z)=x^2y^2z^2.$$

The equation of Steiner's surface mentioned above is

$$m(x^2+2xy+y^2)=xy(x+y+z)(x+y-z).$$

† See "Die moderne Dreiecksgeometrie," by Dr. E. Wölffing. *Umschau*, No. 45, vol. 1.

The former, who recently resigned the office of secretary after thirty years' tenure, thus severs his connection with the Council, on which he has served from (practically) its birth in 1865. President ELLIOTT took as the subject of his retiring address "Some secondary needs and opportunities of English mathematicians."

A CONFERENCE on the international catalogue of scientific literature was held recently in London, attended by delegates appointed from eighteen different countries. Professor FELIX KLEIN was the delegate from Germany, and among the representatives from France and Austria were Professors G. DARBOUX and L. BOLTZMANN. Dr. CYRUS ADLER, Librarian of the Smithsonian Institution, represented the United States.

FOR the "Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen" (see BULLETIN, 2d series, vol. 3, p. 326, vol. 4, p. 32, vol. 5, p. 109), projected by the academies of science of Göttingen, Munich, and Vienna, with Professors H. BURKHARDT and W. FR. MEYER as editors-in-chief, the following list of collaborators and subjects is announced:

Volume I: Arithmetic and Algebra, edited by W. FR. MEYER, of Königsberg.—*A.* Arithmetic: 1. Foundations of arithmetic, H. SCHUBERT, of Hamburg; 2. Permutations and combinations, E. NETTO, of Giessen; 3. Irrational numbers and convergence of infinite processes, A. PRINGSHEIM, of Munich; 4. Complex numbers, E. STUDY, of Greifswald; 5. Mengenlehre, A. SCHOENFLIES, of Göttingen; 6. Finite discrete groups, H. BURKHARDT, of Zürich; 7. Numerical reckoning, R. MEHMKE, of Stuttgart.—*B.* Algebra: 1. Foundations: *a*, *b*. Rational functions, E. NETTO, of Giessen; *c*. Algebraic forms, G. LANDSBERG, of Heidelberg; 2. Theory of invariants, W. FR. MEYER, of Königsberg; 3. Equations: *a*. Separation and approximation of the roots, C. RUNGE, of Hanover; *b*. Rational functions of the roots, K. TH. VALEN, of Königsberg; *c*, *d*. Galois' theory and its applications, O. HÖLDER, of Leipzig; *e*. Systems of equations, K. TH. VALEN, of Königsberg; *f*. Finite groups of linear substitutions, P. WIMAN, of Lund.—*C.* Theory of numbers: 1. Elementary theory of numbers, P. BACHMANN, of Weimar; 2. Theory of forms, K. TH. VALEN, of Königsberg; 3. Analytical theory of numbers, P. BACHMANN, of Weimar; 4. Algebraic numbers, D. HILBERT, of Göttingen; 5. Arithmetic theory of algebraic quantities, G. LANDSBERG, of Heidelberg; 6. Complex multipli-

cation, H. WEBER, of Strassburg.—*D.* Probabilities and Compensation of Errors: 1. Calculus of probabilities, E. CZUBER, of Vienna; 2. Calculus of compensations, 3. Interpolation, J. BAUSCHINGER, of Berlin; 4. Applications of the calculus of probabilities, L. V. BORTKEWITSCH, of Strassburg.—*E.* Calculus of differences, D. SELIWANOFF, of St. Petersburg.

Volume II: Analysis, edited by H. BURKHARDT, of Zürich.—*A.* Analysis of Real Quantities: 1. Principles of the infinitesimal calculus, A. PRINGSHEIM, of Munich; 2. Differential and integral calculus, A. Voss, of Würzburg; 3. Definite integrals, G. BRUNEL, of Bordeaux; 4. Ordinary differential equations, P. PAINLEVÉ, of Paris; 5. Partial differential equations, E. v. WEBER, of Munich; 6. Continuous transformation groups, L. MAURER, of Tübingen; 7. Boundary conditions, *a.* Ordinary differential equations, M. BÔCHER, of Cambridge, Mass.; *b.* Partial differential equations of potential theory, H. BURKHARDT, of Zürich, and W. FR. MEYER, of Königsberg; *c.* Other partial differential equations, A. SOMMERFELD, of Clausthal; 8. Development in series, H. BURKHARDT, of Zürich, and A. SOMMERFELD, of Clausthal; 9. Calculus of variations, A. KNESER, of Dorpat.—*B.* Analysis of Complex Quantities: 1. General theory of functions, W. F. OSGOOD, of Cambridge, Mass.; 2. Algebraic functions and their integrals, W. WIRTINGER, of Innsbruck; 3. Definite integrals, 4. Linear differential equations, H. BURKHARDT, of Zürich; 4*b*. Spherical harmonics and similar functions, A. WANGERIN, of Halle; 5. Non-linear differential equations, P. PAINLEVÉ, of Paris; 6. Inverse functions, *a.* Elliptic functions, J. HARKNESS, of Bryn Mawr, Pa.; *b.* Abelian functions, W. WIRTINGER, of Innsbruck; *c.* Automorphic functions, R. FRICKE, of Braunschweig; 7. Theta functions, A. KRAZER, of Strassburg, and W. WIRTINGER, of Innsbruck; 8. Functional equations and operations, S. PINCHERLE, of Bologna.

Volume III: Geometry, edited by W. FR. MEYER, of Königsberg.—*A.* Pure Geometry: 1. Principles of geometry, F. ENRIQUES, of Bologna; 2. Elementary geometry, M. SIMON, of Strassburg in E.; 3. Partitions and configurations of space, H. STEINITZ, of Berlin; 4. Analysis situs, G. BRUNEL, of Bordeaux; 5. Foundations of projective geometry, A. SCHOENFLIES, of Göttingen; 6. Descriptive geometry, C. RODENBERG, of Hanover; 7. Geometry of inversion, H. BURKHARDT, of Zürich. —*B.* Foundations of the Application of Algebra and Anal-

ysis to Geometry: 1. Fundamental questions, 2. Methods of coördinates, A. SCHOENFLIES, of Göttingen; 3. Geometrical analysis, H. BURKHARDT, of Zürich.—*C.* Algebraic Geometry: 1. Conic sections, F. DINGELDEY, of Darmstadt; 2. General theory of higher algebraic curves, K. ROHN, of Dresden; 3. Special algebraic curves, G. KOHN, of Vienna; 4. Surfaces of the second order, O. STAUDE, of Rostock; 5. General theory of higher algebraic surfaces, G. CASTELNUOVO, of Rome, and F. ENRIQUES, of Bologna; 6. Special algebraic surfaces, W. FR. MEYER, of Königsberg; 7. Spaces of higher dimensions, C. SEGRE, of Turin; 8. Algebraic transformations and correspondences, G. CASTELNUOVO, of Rome, and F. ENRIQUES, of Bologna; 9. Geometry of higher space elements, E. WÄLSCH, of Brünn; 10. Enumerative methods, H. G. ZEUTHEN, of Copenhagen. —*D.* Differential Geometry: 1, 2. Applications of the infinitesimal calculus to curves and surfaces, H. v. MANGOLDT, of Aachen; 3. Curves on surfaces, R. v. LILIENTHAL, of Münster; 4. Particular transcendental curves, G. SCHEFFERS, of Darmstadt; 5. Special transcendental surfaces, R. v. LILIENTHAL, of Münster; 6. Development and representation of two surfaces on each other, A. VOSS, of Würzburg; 7. Contact transformations, G. SCHEFFERS, of Darmstadt; 8. Form of curves that are defined by differential equations, W. DYCK, of Münich; 9. Differential line geometry, E. WÄLSCH, of Brünn; 10. Differential geometry of manifoldnesses of higher dimensions, P. STÄCKEL, of Kiel.

The above three volumes constitute the part of the work devoted to pure mathematics. The second part occupied with applied mathematics will be edited, with the coöperation of FELIX KLEIN, by H. BURKHARDT for subjects in astronomy and geophysics, by A. SCHOENFLIES for mechanics, and by A. SOMMERFELD for physics and physical technology. W. FR. MEYER is to prepare a concluding volume containing the questions of history, philosophy and pedagogy, and a general register of the complete work.

THE MATHEMATISCHE ANNALEN has issued an elaborate general register, prepared by Professor A. SOMMERFELD, of Clausthal, of the first fifty volumes of this journal. The register contains over two hundred pages and is accompanied by a portrait of Alfred Clebsch who, together with Professor Carl Neumann, of the University of Leipzig, founded the *Annalen* in 1868. The first fifty volumes contain fifteen hundred and fifty papers written by three hundred and thirty-two authors. Of these papers, four hundred

and eighteen are devoted to arithmetic and algebra, six hundred and twenty-eight to analysis, five hundred and fifty-one to geometry, eighty-six to applied mathematics and seventeen to the history of mathematics. Of the authors, two hundred and nine are from Germany, twenty-seven from Austria, twenty-two from Italy, eighteen from Russia, fifteen from France, nine from Switzerland, seven from England, seven from the United States, five from Norway and Sweden, five from Denmark, four from Holland, one each from Belgium, Greece, Servia, and Spain. These facts are drawn from the extensive table of statistics which concludes the register. The great utility and convenience of this register to mathematicians is indicated by its threefold classification of the material, 1° relative to the authors' names arranged in alphabetical order, 2° with regard to subjects under sixty-seven different headings, 3° volume by volume.

THE last list of announcements of the Cambridge University Press includes the following mathematical works: "Collected mathematical papers," by Professor P. G. TAIT; "The scientific papers of John Couch Adams," volume 2, edited by Professors W. G. ADAMS and R. A. SAMPSON; "A treatise on octonions," a development of Clifford's biquaternions, by Professor ALEXANDER MCAULAY; "On the kinetic theory of gases," by Mr. S. H. BURBURY, F.R.S.; "A treatise on spherical astronomy," by Professor ROBERT S. BALL, F.R.S.; "A treatise on geometrical optics," by Mr. R. A. HERMAN; "A treatise on the dynamics of a particle," by Dr. E. J. ROUTH, F.R.S.

THE second edition of a "Catalogue of scientific and technical periodicals," by Professor H. C. BOLTON has just been published by the Smithsonian Institution.

L'Éducation Mathématique, which made its initial appearance with the beginning of the present academic year, promises to be a worthy addition to the list of excellent elementary mathematical journals now issued in Paris. Under the editorship of Professors J. GRIESS and H. VUIBERT, it is addressed primarily to students of the classes in letters. It is to be published in quarto form twice a month during the period from October to July inclusive and its office of publication is 17 rue des Écoles, Paris.

"Les savants modernes. Leur vie et leurs travaux, d après les documents académiques choisis et abrégés par A. REBIÈRE. Un vol. in 8° orné de 40 portraits" is announced

to be in the press of Nony et C^ie^, of Paris. The volume is to consist of four parts devoted respectively to the academy of sciences, mathematicians and astronomers, physicists and chemists, and naturalists. The mathematicians and astronomers given a place are the following: Cassini, Huygens, Newton, Leibnitz, the Bernouillis, Euler, Clairaut, d'Alembert, Lagrange, Herschel, Monge, Laplace, Delambre, Legendre, Carnot, Fourier, Gauss, Poncelet, Cauchy, Chasles, LeVerrier, while Ampère, Malus and Fresnel are among the names appearing in the third section of the book.

THE Hinrichs press of Leipzig announces a memoir by Professor JAMES A. CRAIG, of the University of Michigan, on the astrological-astronomical tablets of the Kujundjik (Nineveh) collection of the British Museum, known as the Illumination of Bêl. This very important series, numbering about one hundred and thirty tablets, is here edited for the first time.

OXFORD UNIVERSITY. The following courses of lectures in mathematics are offered for the current Michaelmas term :—Professor ELLIOTT: Theory of numbers, two hours; Differential invariants and reciprocants, one hour.—Professor TURNER: Elementary mathematical astronomy, two hours.—Professor ESSON: Analytic theory of plane curves, two hours; Synthetic theory of plane curves, one hour.—Mr. C. E. HASELFOOT: Algebra, two hours.—Mr. A. E. JOLLIFFE: Series and continued fractions, two hours.—Mr. C. LEUDESDORF: Projective geometry (elementary), three hours.—Mr. J. W. RUSSELL: Pure geometry, two hours; Introduction to higher algebra, one hour.—Mr. C. H. SAMPSON: Analytical geometry, two hours.—Mr. E. H. HAYES: Elementary mechanics, two hours.—Mr. A. L. PEDDER: Problems in pure mathematics, two hours.—Mr. J. E. CAMPBELL: Differential equations, three hours.—Mr. C. H. THOMPSON: Dynamics of a particle, two hours.—Mr. A. L. DIXON: Hydrostatics, one hour.—Mr. H. T. GERRANS: Rigid dynamics, two hours.

PROFESSOR EDGAR W. BASS, professor of mathematics in the United States Military Academy at West Point, after a twenty years' tenure of the position, has been placed on the retired list.

DR. ADAM NELL, professor of mathematics in the Darmstadt Technical Institute, has retired at the age of seventy-four years.

PROFESSOR O. HÖLDER, of the University of Königsberg,

has accepted a call to a professorship in mathematics at the University of Leipzig, and will begin lecturing at the latter university in the summer semester of 1899.

DR. R. HAUSSNER has been appointed professor of mathematics at the University of Giessen. PROFESSOR E. WÄLSCH has been promoted to a professorship of mathematics in the Technical Institute at Brünn. Dr. JOSEPH WELLSTEIN has qualified as privat docent in mathematics at the University of Strassburg.

MR. LUTHER M. DEFOE has been made assistant professor of mathematics in Missouri State University. Mr. T. C. ESTEY has been appointed instructor in mathematics in Amherst College. MR. W. L. SNOW and MR. G. M. JONES have been appointed to instructorships in mathematics in Boston University and Oberlin College, respectively.

DR. W. NYLAND has been promoted to a professorship of astronomy in the University of Utrecht.

BY a recent decree of the Belgian government its astronomical service is separated from the meteorological service; PROFESSOR C. LAGRANGE has been made director of the former.

HENRY PERIGAL, ESQ., author of various kinematical and astronomical works, died recently in London at the age of ninety-seven years.

PROFESSOR W. LECONTE STEVENS, of the Rensselaer Polytechnic Institute, has accepted the chair of physics in Washington and Lee University. Professor G. W. PATTERSON, of the department of physics at the University of Michigan, has been granted a year's leave of absence. Mr. C. D. CHILDS has been made professor of physics at Colgate University. The professorship of physics in Union College, vacated by Mr. F. F. THOMPSON to accept a position in Pennsylvania State College, has been filled by the appointment of Professor H. T. EDDY, of the University of Minnesota. Dr. RICHARD WACHSMUTH has been called from the University of Göttingen to an assistant professorship of physics in the University of Rostock. Dr. S. J. BARNETT has been appointed instructor in physics in Colorado College, and Mr. R. R. LAWRENCE, to a similar position in the Massachusetts Institute of Technology.

DR. PAUL TANNERY contributes to the October number of the *Revue Philosophique* a general review of "The theory of mathematical knowledge according to recent works," bas-

ing his interpretation on Laisant's "La mathématique, philosophie, enseignement," Whitehead's "Treatise on universal algebra, with applications," and Russell's "Essay on the foundations of geometry."

It is proposed to erect a suitable memorial to JAMES CLERK MAXWELL, in the parish church of Corsock, of which he was a trustee and elder. Subscriptions will be received by the Rev. George Sturrock, The Manse, Corsock by Dalbeattie, N. B.

MR. CHARLES A. SCHOTT, chief of the computation division of the U. S. Coast and Geodetic Survey, has been awarded the Wilde prize of the French Academy for his work in terrestrial magnetism.

MR. HAROLD HILTON has been given a mathematical fellowship at Magdalen College of Oxford University.

A travelling fellowship in mathematics available for the purpose of study in Germany has been founded at the University of Paris by an anonymous donor.

A NEW class of fellowships has been created this at year Cornell University. Those holding the degree of doctor of philosophy may obtain these fellowships, which carry no emolument, but all university privileges.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALDAUF (G.). Ueber die Punkte kleinster Summe der absoluten Abstände von n Geraden. Plauen, 1898. 4to. 30 pp. M. 1.80

CHEMIN (O.). See SALMON (G.).

CZUBER (E.). Vorlesungen über Differential- und Integralrechnung. Band II: Integralrechnung. Leipzig, Teubner, 1898. 8vo. 9 and 428 pp. Cloth. M. 10.00

FIEDLER (W.). See SALMON (G.).

FORT (O.) und SCHLÖMILCH (O.). Lehrbuch der analytischen Geometrie. Theil II: analytische Geometrie des Raumes, von O. Schlömilch. 6te Auflage von R. Heger. Leipzig, Teubner, 1898. 8vo. 8 and 338 pp. M. 5.00

FUHRMANN (W.). Beiträge zur Transformation algebraisch-trigonometrischer Funktionen. Königsberg, 1898. 4to. 12 pp. M. 1.20

GRÜNFELD (E.). Zur Theorie der algebraischen auflösbaren Gleichungen. Nikolsburg, 1898. 8vo. 11 pp. M. 1.00

GUTZMER (A.). See JAHRESBERICHT, etc.

HEGER (R.). See FORT (O.).

HALL (F.). Die älteren rein geometrischen Beweise zu Steiners Construction der Malfattischen Aufgabe. Wattenscheid, 1898. 4to. 13 pp. M. 1.20

HORN (J.). Ueber das Verhalten der Integrale von Differentialgleichungen bei der Annäherung der Veränderlichen an eine Unbestimmtheitsstelle. Theil III. Berlin, 1898. 4to. 24 pp. M. 2.50

JAHRESBERICHT der deutschen Mathematiker-Vereinigung. Herausgegeben von A. Wangerin und A. Gutzmer. Band V. Leipzig, Teubner, 1898.

LAMBERT (P. A.). Differential and integral calculus for technical schools and colleges. New York, The Macmillan Co., 1898. 12mo. 10 and 245 pp. Cloth. $1.50

LOPIC (P. J.). Das aus den primitiven Wurzeln der Einheit gebildete complexe Zahlensystem. Halle, 1898. 8vo. 52 pp. M. 1.80

MCMAHON (J.) and SNYDER (V.). Elements of the differential calculus. New York, American Book Co., 1898. 8vo. 14 and 337 pp. Cloth. $2.00

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